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ON
Algebra
AND
THEORY OF EQUATIONS

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PREFACE

This book was written to meet the requirements of the B.A. and B.Sc. students of Indian Universities. Its favourable reception indicates that there was a real need for such a book. The treatment of the subject is in keeping with the principles of modern analysis, but is at the same time simple. In dealing with convergence special care has been taken to make the presentation intelligible to the average student without sacrificing much rigour.

In the present edition the portion on Theory of Equations has been completely rewritten. The former single chapter on the subject has been broken up into four chapters and considerable additions have been made in the articles as well as the examples.

The book contains just a little more than the usual course, and no hesitation should be felt in omitting some of the articles. The examples are ample in number, and are well graded. Of these some are original, many have been taken from the examination papers of various universities and the others are such as are common to practically all text-books on the subject. Reference in a problem to some university is not meant to indicate the source of the problem, but merely to indicate the type of questions asked in the various universities.

I am greatly indebted to my father late Dr. Gorakh Prasad, who went through the whole manuscript of the original book as well as the additional chapters on Theory of Equations in the present edition, and offered valuable criticism and suggestions. My thanks are also due to Mr. R.N. Chaudhuri, B.A. (Cantab.), and to Dr. U. N. Singh, D. Phil., for their valuable suggestions; to Dr. H. C. Gupta, Ph. D., for verifying most of the answers; and to Mr. Jagat Narain, M. Sc., for verifying the remaining answers.

University of Roorkee
June, 1961

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CONTENTS

ALGEBRA

CHAPTER I

BINOMIAL THEOREM

1·1. Product of Binomial Factors. The Binomial Theorem is one of the most important theorems in algebra It gives the expansion of $(x+a)^n$, where n is any number whatever. The theorem will be proved here for rational indices only. We shall first find the expansion of $(x+a)^n$ when n is a positive integer, and then deduce the result when n is fractional or negative.

Consider the following products :

$$(a+b)(p+q)=ap+aq+bp+bq,$$

$$(a+b)(p+q)(x+y)=apx+aqx+bpx+bqx$$
$$+apy+aqy+bpy+bqy, \text{ etc.}$$

We see that the continued product of any number of factors of the form $a+b$ is equal to the sum of all the products which can be formed by choosing one term from each factor and multiplying out. This result will be used to obtain the expansion of $(x+a)^n$.

1·2. Binomial Theorem: Positive Integral Index. *To find the expansion of* $(x+a)^n$ *when* n *is a positive integer.*

Wh n n is a positive integer

$$(x+a)^n=(x+a)(x+a)(x+a)\ldots \text{ to } n \text{ factors.}$$

The value of this expression is obtained by choosing one term from each factor, multiplying out, and adding all the products so formed.

Let us choose the term x from each factor. Then the product formed is x^n. Next choose the term a from one of the factors and x from the remaining factors. Then the product formed is ax^{n-1}. As a can be chosen from each one of the n factors in turn, the number of such products in the expansion of $(x+a)^n$ is n and their sum is $nx^{n-1}a$.

Again, we can choose a from any two of the factors and x from the remaining $n-2$ factors and get the product a^2x^{n-2}. Since this can be done in ${}^nC_2$ ways, the sum of such products is ${}^nC_2x^{n-2}a^2$.

In general, we can choose a from any r factors and x from the remaining $n-r$ factors, getting the product a^rx^{n-r}. This can be done in ${}^nC_r$ ways. So a^rx^{n-r} will be repeated ${}^nC_r$ times.

Therefore the expansion for $(x+a)^n$ is

$$x^n+{}^nC_1x^{n-1}a+{}^nC_2x^{n-2}a^2+\ldots+{}^nC_rx^{n-r}a^r+\ldots+{}^nC_na^n.$$

Substituting in it the values of ${}^nC_1$, ${}^nC_2$, ... we get

$$\mathbf{(x+a)^n=x^n+nx^{n-1}a+\frac{n(n-1)}{2!}x^{n-2}a^2+\ldots}$$

$$\mathbf{+\frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^{n-r}a^r+\ldots+a^n.}$$

This is the *Binomial Theorem* for a positive integral index

Ex. Expand $(2-\frac{3}{2}x^2)^4$.

$$(2-\tfrac{3}{2}x^2)^4=(2)^4+4(2)^3(-\tfrac{3}{2}x^2)+\frac{4.3}{1.2}(2)^2(-\tfrac{3}{2}x^2)^2$$
$$+\frac{4.3.2}{1.2.3}(2)(-\tfrac{3}{2}x^2)^3+(-\tfrac{3}{2}x^2)^4$$
$$=16-48x^2+54x^4-27x^6+\tfrac{81}{16}x^8.$$

1·21. Properties of the Binomial Expansion. (i) The expansion of $(x+a)^n$ contains $n+1$ terms. The $(r+1)$th term is

$${}^nC_r x^{n-r} a^r.$$

This is usually called the *general term.*

(ii) In the expansion of $(x+a)^n$ the terms ${}^nC_r x^{n-r} a^r$ and ${}^nC_{n-r} x^r a^{n-r}$ are equidistant from the beginning and the end. For ${}^nC_r x^{n-r} a^r$ is preceded by r terms and followed by $n-r$ terms, while ${}^nC_{n-r} x^r a^{n-r}$ is preceded by $n-r$ terms and followed by r terms. Also

$${}^nC_{n-r}=n!/(n-r)!\,r!={}^nC_r.$$

Therefore, the coefficients of terms equidistant from the beginning and the end are equal.

(iii) If n is even, the number of terms is odd, and there is one middle term. If n is odd, the number of terms is even and there are two middle terms.

(iv) By writing $-a$ for a, we get

$$(x-a)^n=x^n-{}^nC_1x^{n-1}a+{}^nC_2x^{n-2}a^2-{}^nC_3x^{n-3}a^3+\dots+(-1)^na^n.$$

It should be noticed that here the terms are alternately positive and negative.

(v) By writing $(a+x)^n$ instead of $(x+a)^n$, we get

$$(a+x)^n=a^n+{}^nC_1a^{n-1}x+{}^nC_2a^{n-2}x^2+\dots+{}^nC_ra^{n-r}x^r+\dots+x^n.$$

Putting $a=1$ in this, we get

$$(1+x)^n=1+{}^nC_1x+{}^nC_2x^2+\dots+{}^nC_rx^r+\dots+x^n.$$

i.e.,
$$\mathbf{(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\dots}$$
$$\mathbf{+\frac{n(n-1)\dots(n-r+1)}{r!}x^r+\dots+x^n.} \quad (1)$$

We shall consider this as the standard form of the Binomial Theorem. This form is simpler, and will be found useful later on. Since

$$(a+x)^n = a^n(1+x/a)^n,$$

the expansion of $(a+x)^n$ can be always deduced from (1).

Ex. 1. Find the middle term in the expansion of

$$\left(\frac{a}{x}+\frac{x}{a}\right)^{10}.$$

The number of terms is 11; so the middle term is the 6th. Hence the middle term $= {}^{10}C_5(a/x)^5(x/a)^5 = 252$.

Ex. 2. Find the coefficient of x^3 in the expansion of $(x-1/x)^7$.

Since $\quad (x-1/x)^7 = x^7(1-1/x^2)^7 = x^7(1-x^{-2})^7,$

the coefficient of x^3 in $(x-1/x)^7$ is the same as the coefficient of x^{-4} in $(1-x^{-2})^7$. Hence the required coefficient

$$= {}^7C_2(-1)^2 = 21.$$

1·22. Binomial Coefficients. The coefficients in the expansion of $(1+x)^n$, when n is a positive integer, are known as *binomial coefficients*. They are generally denoted by $c_0, c_1, c_2, \ldots, c_n$. Thus $c_r = {}^nC_r$, and

$$(1+x)^n = c_0 + c_1x + c_2x^2 + \ldots + c_nx^n. \qquad (1)$$

(i) *To find the sum of the binomial coefficients.*

Put $x=1$ in (1); then

$$c_0 + c_1 + c_2 + \ldots + c_n = 2^n.$$

(ii) *The sum of the odd coefficients is equal to the sum of the even coefficients in the binomial expansion.*

Put $x=-1$ in (1); then

$$c_0 - c_1 + c_2 - c_3 + c_4 - c_5 + \ldots = 0,$$

i.e., $\quad c_0 + c_2 + c_4 + \ldots = c_1 + c_3 + c_5 + \ldots .$

Ex. Show that $c_1+2c_2x+3c_3x^2+\ldots+nc_nx^{n-1}=n(1+x)^{n-1}$, and deduce the value of $c_1{}^2+2^2c_2{}^2+3^2c_3{}^2+\ldots+n^2c_n{}^2$.

We have $\quad c_1+2c_2x+3c_3x^2+\ldots+nc_nx^{n-1}$

$$=n+2\frac{n(n-1)}{2!}x+3\frac{n(n-1)(n-2)}{3!}x^2+\ldots+nx^{n-1}$$

$$=n\left\{1+(n-1)x+\frac{(n-1)(n-2)}{2!}x^2+\ldots+x^{n-1}\right\}=n(1+x)^{n-1}.$$

Thus $c_1+2c_2x+3c_3x^2+\ldots+nc_nx^{n-1}=n(1+x)^{n-1}$. (1)

Again, putting $1/x$ for x in this, we get

$$c_1+\frac{2c_2}{x}+\frac{3c_3}{x^2}+\ldots+\frac{nc_n}{x^{n-1}}=n\left(1+\frac{1}{x}\right)^{n-1}. \tag{2}$$

Multiplying (1) and (2), and collecting, on the left-hand side, the terms free from x, we get

$$c_1{}^2+2^2c_2{}^2+3^2c_3{}^2+\ldots+n^2c_n{}^2. \tag{3}$$

But the product of the right-hand sides is

$$n^2(1+x)^{n-1}(1+1/x)^{n-1}=(n^2/x^{n-1})(1+x)^{2n-2},$$

and the term free from x in it is

$$(n^2/x^{n-1})\,{}^{2n-2}C_{n-1}\,x^{n-1},\text{ i.e., } n^2\,.\,(2n-2)!/(n-1)!(n-1)!$$

This, therefore, must be the value of (3).

EXAMPLES

Expand the following expressions :

1. $(3x-2y)^4$.
2. $(x^2+x)^5$.
3. $(\frac{2}{3}x-3/2x)^6$.
4. $(1+2x-x^2)^4$.
5. Simplify $(\sqrt{2}+1)^5-(\sqrt{2}-1)^5$.
6. Find the 3rd term of $(2x-\frac{1}{2})^6$.
7. Find the 13th term of $(9x-1/3\sqrt{x})^{18}$.
8. Find the middle term of $(\frac{1}{3}a+9b)^8$.
9. Find the two middle terms of $(x^4-1/x^3)^{11}$.
10. Find the coefficient of x^9 in $(x+3a/x^2)^{15}$.
11. Find the term independent of x in $(\frac{3}{2}x^2-1/3x)^9$.

If $(1+x)^n = c_0 + c_1x + c_2x^2 + \ldots + c_nx^n$, prove that

12. $c_0 + 3c_1 + 5c_2 + \ldots + (2n+1)c_n = (n+1)2^n$.

13. $2c_0 + \dfrac{2^2c_1}{2} + \dfrac{2^3c_2}{3} + \dfrac{2^4c_3}{4} + \ldots + \dfrac{2^{n+1}c_n}{n+1} = \dfrac{3^{n+1}-1}{n+1}$.

14. $c_1^2 + 2c_2^2 + 3c_3^2 + \ldots + nc_n^2 = (2n-1)!/\{(n-1)!\}^2$.

15. $c_0c_1 + c_1c_2 + c_2c_3 + \ldots + c_{n-1}c_n = (2n)!/(n+1)!(n-1)!$

1·3. Binomial Series. When n is a positive integer the series

$$1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \quad (1)$$

terminates after $n+1$ terms and is the expansion of $(1+x)^n$. But when n is not a positive integer the series does not terminate and every term is succeeded by another term. In fact, we have here what is known as an infinite series. An example is

$$1 - \tfrac{1}{2}x + \frac{1.3}{2.4}x^2 + - \ldots + (-1)^r \frac{1.3.5\ldots(2r-1)}{2.4.6\ldots2r}x^r + \ldots,$$

obtained by putting $n = -\frac{1}{2}$ in (1).

Infinite series do not always have a meaning; and the algebraical operations of addition and multiplication cannot always be performed upon them. In order that an infinite series may have a meaning it must have a definite sum; that is, if we add up the first r terms, the resulting sum must approach a definite value as r increases.

In this sense the geometric series $1 + \frac{1}{2} + (\frac{1}{2})^2 + (\frac{1}{2})^3 + \ldots$ has a perfectly definite meaning. The sum of the first r terms of the series is $2 - (\frac{1}{2})^{r-1}$, which approaches a definite value (2 in this case) as r increases. Series which possess this property are called convergent series (see Chap. VII). It can be shown that the series (1) is convergent if x is numerically less than unity.

When a convergent series contains both positive and negative terms, a new series can be obtained by making the sign of all the terms positive. If this series is also convergent, the original series is called absolutely convergent. It can be shown that (1) is also absolutely convergent when x is numerically less than unity (§ 7·71).

We shall prove in the following articles that series (1) is the expansion of $(1+x)^n$ even when n is fractional or negative. The proof involves the multiplication of series (1) by itself. It will be shown in a later chapter that multiplication of two infinite series is valid if they are absolutely convergent (§ 7·82). Therefore, our proof will be valid for values of x numerically less than unity.

When x is numerically greater than unity, the series (1) does not have a meaning. The sum to r terms either gets larger and larger as r increases, or is alternately positive and negative while numerically it goes on increasing. In neither case the series can be said to have a sum, and the question of its equality to $(1+x)^n$ has no meaning.

When $x=1$ or -1 the series (1) requires a closer examination.

1·4. Vandermonde's Theorem. In the present article and the next one we shall use $x_{(r)}$ to denote the product

$$x(x-1)(x-2)\ldots(x-r+1);$$

thus, $x_{(1)}=x$, $x_{(2)}=x(x-1)$, etc. Here x is not necessarily a positive integer; it may be any number whatever.

Vandermonde's theorem states that *if m and n are any two numbers, then*

$$(m+n)_{(r)}=m_{(r)}+{}^rC_1 m_{(r-1)}n_{(1)}+{}^rC_2 m_{(r-2)}n_{(2)}+\ldots+n_{(r)}.$$

Suppose first that m and n are positive integers. Then, by the Binomial Theorem,

$$(1+x)^m=1+\frac{m_{(1)}}{1!}x+\frac{m_{(2)}}{2!}x^2+\ldots+\frac{m_{(r)}}{r!}x^r+\ldots,$$

and $$(1+x)^n=1+\frac{n_{(1)}}{1!}x+\frac{n_{(2)}}{2!}x^2+\ldots+\frac{n_{(r)}}{r!}x^r+\ldots.$$

Multiplying the respective sides, we get $(1+x)^{m+n}$ on the left and a series which can be arranged in ascending powers of x on the right. Equating the coefficients of x^r in these two, we get

$$\frac{(m+n)_{(r)}}{r!}=\frac{m_{(r)}}{r!}+\frac{m_{(r-1)}n_{(1)}}{(r-1)!\,1!}+\frac{m_{(r-2)}n_{(2)}}{(r-2)!\,2!}+\ldots+\frac{n_{(r)}}{r!}.$$

Multiplying this by $r!$ we get the result

$$(m+n)_{(r)}=m_{(r)}+{}^rC_1m_{(r-1)}n_{(1)}+{}^rC_2m_{(r-2)}n_{(2)}+\ldots+n_{(r)}. \quad (1)$$

This holds for all positive integral values of m and n. But each side is a finite expression of the same degree in m and n. Therefore (1) must be an identity, that is, if the product $(m+n)_{(r)}$ is written in full and arranged in powers of m, say, then each term must be equal to a corresponding term on the other side. This being so, the equation (1) must be true for *all* values of m and n.

1·5. Binomial Theorem: Any Index. *If n is any number, positive or negative, integral or fractional, and x is numerically less than unity, the sum of the series*

$$1+nx+\frac{n(n-1)}{2!}x^2+\ldots+\frac{n(n-1)\ldots(n-r+1)}{r!}x^r+\ldots$$

is $(1+x)^n$.

When x is numerically less than unity, the given series is absolutely convergent for all values of n (see § 7·71). Denote its sum by $f(n)$, and let

$$n_{(r)}=n(n-1)(n-2)\ldots(n-r+1).$$

Then

$$f(n)=1+\frac{n_{(1)}}{1!}x+\frac{n_{(2)}}{2!}x^2+\ldots+\frac{n_{(r)}}{r!}x^r+\ldots,$$

and $$f(m)=1+\frac{m_{(1)}}{1!}x+\frac{m_{(2)}}{2!}x^2+\ldots+\frac{m_{(r)}}{r!}x^r+\ldots.$$

Multiplying these and arranging in powers of x, we get

$$f(m)\,.\,f(n)=1+k_1x+k_2x^2+\ldots+k_rx^r+\ldots,$$

where $$k_r=\frac{m_{(r)}}{r!}+\frac{m_{(r-1)}n_{(1)}}{(r-1)!\,1!}+\frac{m_{(r-2)}n_{(2)}}{(r-2)!\,2!}+\ldots+\frac{n_{(r)}}{r!}$$

$$=\frac{(m+n)_{(r)}}{r!}, \text{ by Vandermonde's theorem.}$$

Therefore $f(m)\,.\,f(n)=1+\dfrac{(m+n)_{(1)}}{1!}x+\dfrac{(m+n)_{(2)}}{2!}x^2$

$$+\ldots+\frac{(m+n)_{(r)}}{r!}x^r+\ldots,$$

i.e., $$f(m)\,.\,f(n)=f(m+n). \qquad (1)$$

Again, $f(m)\,.\,f(n)\,.\,f(p)=f(m+n)\,.\,f(p)=f(m+n+p)$. Repeated applications will show that the result holds for any number of factors.

CASE I. *Positive fractional index.* Let $n=p/q$, where p and q are positive integers. Then, by what has been shown above,

$$f(p/q)\,.\,f(p/q)\ldots \text{ to } q \text{ factors}=f(q\,.\,p/q)=f(p).$$

But $f(p)=1+px+\dfrac{p(p-1)}{2!}x^2+\ldots=(1+x)^p$,

since p is a positive integer. Therefore

$$\{f(p/q)\}^q=(1+x)^p,$$

or, taking the qth root of both the sides,

$$f(p/q)=(1+x)^{p/q}.$$

CASE II. *Negative index.* Let $n=-m$, where m is any positive integer or fraction; then by (1)

$$f(m)\,.\,f(-m)=f(0)=1.$$

But $f(m)=(1+x)^m$ by Case I. Therefore

$$f(-m)=1/f(m)=1/(1+x)^m=(1+x)^{-m}.$$

Therefore for all values of n, $f(n)=(1+x)^n$, i.e.,

$$\mathbf{(1+x)^n=1+nx+\frac{n(n-1)}{2!}x^2+\ldots}$$

$$\mathbf{+\frac{n(n-1)\ldots(n-r+1)}{r!}x^r+\ldots.} \qquad (2)$$

NOTE. (i) If n is fractional, $(1+x)^n$ will have more than one value, e.g. $(1+\frac{7}{9})^{1/2}=\pm\frac{4}{3}$. The sum of the series in (2) is the real positive value of $(1+x)^n$. This can be seen by putting $x=0$ in (2).

(ii) A number n can be expressed as a ratio of two integers only when n is rational. So the proof given above is for rational indices only. For irrational indices the theorem follows from the continuity of the function represented by the series (2). But the actual proof is beyond the scope of this book.

1·51. Particular Cases. (i) Consider the expansion of $(1-x)^{-n}$. The general term

$$=\frac{(-n)(-n-1)(-n-2)\ldots(-n-r+1)}{r!}(-x)^r$$

$$=(-1)^{2r}\frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}x^r.$$

Therefore

$$(1-x)^{-n}=1+nx+\frac{n(n+1)}{2!}x^2+\ldots$$

$$+\frac{n(n+1)\ldots(n+r-1)}{r!}x^r+\ldots. \quad (1)$$

The expansion of $(1+x)^{-n}$ is similar, only the terms are alternately positive and negative.

(ii) By giving to n the values 1 and 2, we obtain from (1)

$$(1-x)^{-1}=1+x+x^2+x^3+\ldots+x^r+\ldots,$$

$$(1+x)^{-1}=1-x+x^2-x^3+\ldots+(-1)^r x^r+\ldots,$$

$$(1-x)^{-2}=1+2x+3x^2+4x^3+\ldots+(r+1)x^r+\ldots,$$

$$(1+x)^{-2}=1-2x+3x^2-4x^3+\ldots+(-1)^r(r+1)x^r+\ldots.$$

The above expansions are frequently used and the student should be familar with them.

(iii) An expression of the form $(x+a)^n$ can be expanded in the following way:

$$(x+a)^n=a^n\left(1+\frac{x}{a}\right)^n=a^n\left\{1+n\frac{x}{a}+\frac{n(n-1)}{2!}\frac{x^2}{a^2}+\ldots\right\}.$$

The expansion is valid for $(x/a)<1$, i.e., $x<a$. For $x>a$, $(x+a)^n$ can be expanded in *descending* powers of x. Thus,

$$(x+a)^n=x^n\left(1+\frac{a}{x}\right)^n=x^n\left\{1+n\frac{a}{x}+\frac{n(n-1)}{2!}\frac{a^2}{x^2}+\ldots\right\}.$$

Ex. Find the general term in the expansion of

$$(1+2x)^{1/2}.$$

The general term$=\dfrac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)\ldots(\frac{1}{2}-r+1)}{r!}(2x)^r$

$$=\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\ldots(-r+\frac{3}{2})}{r!}2^r x^r$$

$$=(-1)^{r-1}\frac{1\,.\,3\,.\,5\ldots(2r-3)}{r!}x^r.$$

1·52. Summation of Series. If a given series can be reduced to the form of a binomial expansion, its sum can be written down directly by the Binomial Theorem.

In reducing the given series the numerator and the denominator of the general term should be multiplied by suitable factors, so that the denominator becomes $r!$ and the numerator contains r factors differing successively from each other by unity, besides the rth power of some number x.

Ex. 1. Find the sum of the series

$$1-\frac{1}{5}+\frac{1.4}{5.10}-\frac{1.4.7}{5.10.15}+\frac{1.4.7.10}{5.10.15.20}-\ldots$$ [*Bombay*, 1947]

This series can be written as

$$1-\tfrac{1}{3}(\tfrac{3}{5})+\frac{(\frac{1}{3})(\frac{4}{3})}{1.2}(\tfrac{3}{5})^2-\frac{(\frac{1}{3})(\frac{4}{3})(\frac{7}{3})}{1.2.3}(\tfrac{3}{5})^3+\frac{(\frac{1}{3})(\frac{4}{3})(\frac{7}{3})(\frac{10}{3})}{1.2.3.4}(\tfrac{3}{5})^4-\ldots$$

$$=1+(-\tfrac{1}{3})(\tfrac{3}{5})+\frac{(-\frac{1}{3})(-\frac{4}{3})}{2!}(\tfrac{3}{5})^2+\frac{(-\frac{1}{3})(-\frac{4}{3})(-\frac{7}{3})}{3!}(\tfrac{3}{5})^3+\ldots.$$

Therefore, its sum

$$=(1+\tfrac{3}{5})^{-1/3}=(\tfrac{8}{5})^{-1/3}=(\tfrac{5}{8})^{1/3}=(\sqrt[3]{5})/2.$$

Ex. 2. Sum the series

$$(\tfrac{1}{2})^2+\frac{1}{2!}(\tfrac{1}{2})^4+\frac{1\,.\,3}{3!}(\tfrac{1}{2})^6+\frac{1\,.\,3\,.\,5}{4!}(\tfrac{1}{2})^8+\ldots.$$

The fourth term $=\dfrac{1.3.5}{4!}(\tfrac{1}{2})^8=\dfrac{(\frac{1}{2})(\frac{3}{2})(\frac{5}{2})}{4!}(\tfrac{1}{2})^5.$

$$=-\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{4!}(\tfrac{1}{2})^5=-\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{4!}(\tfrac{1}{2})^4.$$

Arranging the other terms similarly, the given series

$$=\tfrac{1}{2}(\tfrac{1}{2})-\frac{(\frac{1}{2})(-\frac{1}{2})}{2!}(\tfrac{1}{2})^2+\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}(\tfrac{1}{2})^3$$
$$-\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{4!}(\tfrac{1}{2})^4+\ldots$$

$$=1-\{1-\tfrac{1}{2}(\tfrac{1}{2})+\frac{(\frac{1}{2})(-\frac{1}{2})}{2!}(\tfrac{1}{2})^2-\frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}(\tfrac{1}{2})^3+\ldots\}$$
$$=1-(1-\tfrac{1}{2})^{1/2}=1-1/\sqrt{2}.$$

Examples

Expand to 4 terms the following expressions in ascending powers of x, and state the values of x for which the expansions are valid.

1. $(1+x)^{1/2}$. 2. $(1-nx)^{-1/n}$.

3. $(8+12x)^{2/3}$. 4. $(4a-8x)^{-3/2}$.

5. Find the 11th term in the expansion of $(1-2x^3)^{11/2}$.

Find the general term in the expansions of

6. $(1-3x)^{1/3}$. 7. $(1-x)^{-3}$.

8. $(1+x)^{-1/2}$. 9. $(1-x)^{-1}-2(1-2x)^{-2}$.

10. The expression $a+b/(1+2x)+c/(1-3x^2)$ can be expanded in the $1+x+2x^2+\ldots$. Find a, b and c.

11. Find the coefficient of x^r in the expansion of
$$(2x+1)/(1+x^2).$$

12. Show that the middle term of $(x+1/x)^{4n}$ is equal to the coefficient of x^n in the expansion of $(1-4x)^{-n-1/2}$. [*Bihar*,'54]

13. Use the product $(1-x)^{-1} . (1-x)^{1/2}$ to find the value of $1+\frac{1}{2}+\frac{1.3}{2.4}+\frac{1.3.5}{2.4.6}+\ldots+\frac{1.3.5.\ldots(2n-3)}{2.4.6.\ldots(2n-2)}$. [*Annam.*, '54]

Find the sum of the following series :

14. $1+\frac{2}{3}\cdot\frac{1}{2}+\frac{2.5}{3.6}\cdot\frac{1}{2^2}+\frac{2.5.8}{3.6.9}\cdot\frac{1}{2^3}+\ldots.$ [*Utkal*, 1949]

15. $1+\frac{2}{9}+\frac{2.5}{9.18}+\frac{2.5.8}{9.18.27}+\ldots.$ [*Bombay*, 1952]

16. $1+\frac{5}{8}+\frac{5.8}{8.12}+\frac{5.8.11}{8.12.16}+\ldots.$ [*Annamalai*, 1949]

17. $\frac{3}{2.4}+\frac{3.4}{2.4.6}+\frac{3.4.5}{2.4.6.8}+\ldots.$ [*Poona*, 1959]

18. $1+\frac{1.3}{2.4}x^2+\frac{1.3.5.7}{2.4.6.8}x^4+\ldots,\ |x|<1.$

1·6. Greatest Term. *To find the numerically greatest term in the expansion of* $(1+x)^n$, *where* $|x|<1$.

Since we are concerned with the numerical values of the terms, we shall consider only the positive value of x.

Case I. n positive.

Let u_r denote the rth term; then

$$u_{r+1}=\frac{n-r+1}{r}\,x\,.\,u_r=\left(\frac{n+1}{r}-1\right)x\,.\,u_r. \quad (1)$$

Therefore, $u_{r+1}>,\ =,$ or $<u_r$ according as

$$\left(\frac{n+1}{r}-1\right)x>,\ =,\ \text{or}\ <1,$$

i.e.,
$$(n+1)x>,\ =,\ \text{or}\ <(1+x)r,$$

i.e., $$r <, =, \text{ or } > \frac{(n+1)x}{1+x}.$$

Two cases arise :

(i) $(n+1)x/(1+x)$ is an integer, equal to p, say. Then, as long as $r<p$, $u_{r+1}>u_r$ and the terms go on increasing. When $r=p$, $u_{r+1}=u_r$. When $r>p$, $u_{r+1}<u_r$ and the terms decrease. Therefore the pth and the $(p+1)$th terms are equal and are the greatest terms.

(ii) $(n+1)x/(1+x)$ is equal to an integer p plus a positive proper fraction f, say. In this case as long as $r<p+f$, $u_{r+1}>u_r$; and when $r>p+f$, $u_{r+1}<u_r$. Therefore, for $r \leqslant p$ the terms go on increasing, u_{p+1} being greater than u_p and all the preceding terms. For $r \geqslant p+1$, $u_{r+1}<u_r$ and the terms decrease. Thus u_{p+1} is greater than u_{p+2} and all the succeeding terms. Hence the $(p+1)$th term is the greatest term.

CASE II. n negative.

Let $n=-m$, so that m is positive; then, by (1),

$$u_{r+1}=\frac{-m-r+1}{r}x \,.\, u_r = -\left(\frac{m-1}{r}+1\right)x \,.\, u_r. \quad (2)$$

Therefore, u_{r+1} will be numerically greater than, equal to, or less than u_r, according as

$$\left(\frac{m-1}{r}+1\right)x >, =, \text{ or } < 1,$$

i.e., $$(m-1)x >, =, \text{ or } < (1-x)r,$$

i.e., $$r <, =, \text{ or } > \frac{(m-1)x}{1-x}.$$

Let $m>1$; then, arguing as before, we see that (i) if $(m-1)x/(1-x)=$an integer p, the pth and $(p+1)$th terms are equal and are the greatest

terms; (ii) if $(m-1)x/(1-x)=p+f$, where p is an integer and f is a positive proper fraction, $(p+1)$th term is the greatest term.

If $m<1$, then $(m-1)x/(1-x)$ is negative and the above arguments do not apply. But in this case the factor $\{(m-1)/r+1\}x$, i.e., $\{1-(1-m)/r\}x$, in (2), is less than unity for all values of r. Therefore the first term in the expansion is the greatest term.

Note. (i) When n is positive, the factor $\{(n+1)/r-1\}$ in (1) becomes negative, for $r>n+1$. But its numerical value remains less than unity. Therefore the terms go on decreasing numerically and no amendment to the proof is necessary.

(ii) When x is negative, equal to $-y$ say, we can apply the results of the above article by putting y for x in the formulae.

(iii) When n is a positive integer the expansion is finite; so the restriction that $|x|<1$ can be removed.

(iv) In application to examples it is better to proceed as in the above article, rather than to apply the formulae obtained.

Ex. Find the numerically greatest term in the expansion of $(1+x)^{21/2}$ when $x=\frac{2}{3}$.

We have $u_{r+1}=\dfrac{\frac{21}{2}-r+1}{r}\,\frac{2}{3}u_r=\dfrac{23-2r}{3r}\,u_r.$

Therefore $u_{r+1}>u_r$ as long as $(23-2r)/3r>1$,

i.e., $\qquad 23-2r>3r$ or $r<4\frac{3}{5}$.

Hence the 5th term is the greatest term.

1·7. Application to Approximations. The Binomial Theorem can be used to obtain approximate values, as is illustrated in the following examples.

Ex. 1. Find to five decimal places the value of $\sqrt{(98)}$.

$\sqrt{(98)}=\sqrt{(100-2)}=10\sqrt{(1-\frac{2}{100})}=10(1-\cdot 02)^{1/2}$

$$=10\{1-\tfrac{1}{2}(\cdot 02)+\frac{\frac{1}{2}(-\frac{1}{2})}{1.2}(\cdot 02)^2-\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{1.2.3}(\cdot 02)^3+\ldots\}$$

$$=10\{1-\cdot01-\cdot00005-\cdot0000005-\ldots\}$$
$$=9\cdot89949, \text{ approximately.}$$

The terms which have been neglected do not affect the fifth decimal place.

Ex. 2. Calculate $\sqrt[3]{2}$ to four places of decimals.

We have first to express $\sqrt[3]{2}$ in a form convenient for expansion. The cubes of natural numbers are 1, 8, 27, 64, 125, ... We notice that 125/64 is nearly equal to 2. Therefore

$$\sqrt[3]{2}=\left(\frac{125}{64}+\frac{3}{64}\right)^{1/3}=\tfrac{5}{4}\left(1+\frac{3}{125}\right)^{1/3}$$
$$=\tfrac{5}{4}\left\{1+\tfrac{1}{3}\left(\frac{3}{125}\right)+\frac{\frac{1}{3}(-\frac{2}{3})}{1.2}\left(\frac{3}{125}\right)^2+\ldots\right\}$$
$$=1\cdot25+0\cdot01-0\cdot00008+\ldots=1\cdot2599.$$

EXAMPLES

Which is the greatest term in the following expansions ?

1. $(1+x)^{13/2}$ when $x=\frac{3}{4}$.
2. $(a+x)^9$ when $a=\frac{1}{2}$, $x=\frac{1}{3}$.
3. $(1+x)^{-7}$ when $x=\frac{4}{15}$.
4. $(1+x)^{-1/2}$ when $x=\frac{1}{2}$.
5. $(5-4x)^{-7}$ when $x=\frac{1}{2}$.

6. If x is so small that its square and higher powers can be neglected, find the value of

(i) $\dfrac{(8+3x)^{2/3}}{(2+3x)\sqrt{(4-5x)}}$; [*Mad.*,'60] (ii) $\dfrac{(1-2x)^{2/3}+(1+3x)^{3/4}}{(9+2x)^{1/2}-(8-3x)^{1/3}}$.

Find to five decimal places the value of

7. $\sqrt[3]{(998)}$. 8. $(630)^{-1/4}$.

9. $\sqrt{2}$. 10. $\sqrt[4]{5}$.

11. Show that the value of $\sqrt{(x^2+4)}-\sqrt{(x^2+1)}$ is approximately $1-\frac{1}{4}x^2+\frac{7}{64}x^4$ when x is small, and

$$\frac{3}{2x}\left(1-\frac{5}{4x^2}+\frac{21}{8x^4}\right)$$

when x is large.

12. If x is small so that x^2 and higher powers of x can be neglected, show that the nth root of $1+x$ is equal to

$$\frac{2n+(n+1)x}{2n+(n-1)x} \text{ nearly.} \qquad [\textit{Travancore}, 1953]$$

EXAMPLES ON CHAPTER I

1. Show that the middle term in the expansion of $(1+x)^{2n}$ is

$$\frac{1.3.5\ldots(2n-1)}{n!}2^n x^n.$$

2. Find the term independent of x in the expansion of

$$(x-1/x^2)^{3n}. \qquad [\textit{U.P.F.S.}, 1960]$$

3. Prove that the coefficient of x^r in the expansion of $(1-4x)^{-1/2}$ is $(2r)!/(r!)^2$.

4. If $x>0$, prove that

$$(1+x)^n=2^n\left\{1-n\frac{1-x}{1+x}+\frac{n(n+1)}{2!}\left(\frac{1-x}{1+x}\right)^2-\ldots\right\}.$$

5. If $x>-\frac{1}{2}$, prove that

$$\frac{x}{\sqrt{(1+x)}}=\frac{x}{1+x}+\frac{1}{2}\left(\frac{x}{1+x}\right)^2+\frac{1.3}{2.4}\left(\frac{x}{1+x}\right)^3$$
$$+\frac{1.3.5}{2.4.6}\left(\frac{x}{1+x}\right)^4+\ldots. \qquad [\textit{Karnatak}, \text{'54}]$$

6. Show that

$$1+\frac{n}{2}+\frac{n(n-1)}{2.4}+\frac{n(n-1)(n-2)}{2.4.6}+\ldots$$
$$=1+\frac{n}{3}+\frac{n(n+1)}{3.6}+\frac{n(n+1)(n+2)}{3.6.9}+\ldots.$$

[*Utkal*, 1952]

7. If n is any positive integer, show that the integral part of $(3+\sqrt{7})^n$ is an odd number. [*Bombay*, 1948]

[Hint. On expansion we see that $(3+\sqrt{7})^n+(3-\sqrt{7})^n$ $=$an even integer. Also $(3-\sqrt{7})^n$ is less than unity; hence etc.]

8. Find the first three terms of the expansion, in powers of x, of $(1+3x)^{1/2}(1-2x)^{-1/3}$. [*Madras*, 1948]

9. Expand $(1-x+x^2)^{1/2}$ in ascending powers of x up to the term in x^3.

10. Prove that the coefficient of x^n in the expansion of $1/(1+x+x^2)$ is 1, 0, or -1 according as n is of the form $3m$, $3m-1$, or $3m+1$. [*Patna*, 1953]

11. Show that the coefficient of x^m in the expansion of $(1+2x)/(1-x+x^2)$ is $(-1)^{m/3}$, $3(-1)^{(m-1)/3}$, or $2(-1)^{(m-2)/3}$ according as m is of the form $3n$, $3n+1$, or $3n+2$, where n is a positive integer. [*Delhi*, 1949]

If $(1+x)^n=c_0+c_1x+c_2x^2+\ldots+c_nx^n$, prove that

12. $c_0+2c_1+3c_2+\ldots+(n+1)c_n=2^{n-1}(n+2)$. [*Rangoon*, 1950]

13. $c_0+\frac{1}{2}c_1+\frac{1}{3}c_2+\ldots+c_n/(n+1)=(2^{n+1}-1)/(n+1)$. [*Delhi*, 1950]

14. $c_0^2+c_1^2+c_2^2+\ldots+c_n^2=(2n)!/n!n!$ [*I. A. S.*, 1955]

15. $c_0^2-c_1^2+c_2^2-\ldots+(-1)^nc_n^2=(-1)^{n/2}n!/(\frac{1}{2}n)!(\frac{1}{2}n)!$ or 0, according as n is even or odd.

16. $\dfrac{c_1}{c_0}+\dfrac{2c_2}{c_1}+\dfrac{3c_3}{c_2}+\ldots+\dfrac{nc_n}{c_{n-1}}=\frac{1}{2}n(n+1)$.

17. If n be an even integer, prove that

$$\frac{1}{1!(n-1)!}+\frac{1}{3!(n-3)!}+\frac{1}{5!(n-5)!}+\ldots+\frac{1}{(n-1)!1!}=\frac{2^{n-1}}{n!}.$$

[Hint. Multiply by $n!$; then the left-hand side is merely $c_1+c_3+c_5+\ldots+c_{n-1}$.]

18. If s_n denote the sum of the first n natural numbers, show that

$$s_1+s_2x+s_3x^2+\ldots+s_nx^{n-1}+\ldots=(1-x)^{-3}.$$

Find the sum of the following series :

19. $1-\dfrac{3}{4}+\dfrac{3\,.\,5}{4\,.\,8}-\dfrac{3\,.\,5\,.\,7}{4\,.\,8\,.\,12}+\ldots$ [*Gujarat*, 1959]

20. $1+\frac{4}{6}+\frac{4.5}{6.9}+\frac{4.5.6}{6.9.12}+\dots$ [*Andhra*, 1954]

21. $2+\frac{5}{2!3}+\frac{5.7}{3!\,3^2}+\frac{5.7.9}{4!\,3^3}+\dots$ [*Allahabad*, 1946]

22. $\frac{1}{2.4.6}+\frac{1.3}{2.4.6.8}+\frac{1.3.5}{2.4.6.8.10}+\dots$

[*Bombay*, 1947]

23. Prove that

$$1=\frac{1}{4}+\frac{1.3}{4.6}+\frac{1.3.5}{4.6.8}+\dots$$ [*Agra*, 1941]

24. Show that

$$\frac{5}{3.6}+\frac{5.7}{3.6.9}+\frac{5.7.9}{3.6.9.12}+\dots=\tfrac{1}{3}(3\sqrt{3}-2).$$

[*Poona*, 1960]

25. Identifying as a binomial expansion, show that

$$\frac{1.3}{3.6}+\frac{1.3.5}{3.6.9}+\frac{1.3.5.7}{3.6.9.12}+\dots=0{\cdot}4 \text{ nearly.}$$

[*Rajputana*, 1950]

CHAPTER II

EXPONENTIAL & LOGARITHMIC SERIES

2·1. The number e. Just as the number π, denoting the ratio of the circumference of a circle to its diameter has a special importance in trigonometry, similarly the number equal to the sum of the infinite series

$$1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\dots, \tag{1}$$

is of great importance in algebra. This number is denoted by the letter e.

It is easy to see that series (1) has a finite sum; for

$$1+\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}+\frac{1}{1.2.3.4}+\dots$$

$$<1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\dots,$$

i.e., $$<1+1/(1-\tfrac{1}{2}) \text{ or } 3.$$

Thus e is a number less than 3. Its value is 2·7182818..., which can be calculated from (1) to any desired degree of accuracy.

2·2. Exponential Theorem. *For all values of x, positive or negative, integral or fractional, the sum of the series*

$$1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots+\frac{x^r}{r!}+\dots$$

is e^x.

It can be shown (§7·71) that the given series is absolutely convergent for all values of x. Denote its sum by $f(x)$. Then

$$f(x)=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^r}{r!}+\ldots,$$

and $$f(y)=1+\frac{y}{1!}+\frac{y^2}{2!}+\frac{y^3}{3!}+\ldots+\frac{y^r}{r!}+\ldots.$$

Multiplying these, we get

$$f(x)\,.\,f(y)=1+\left(\frac{x}{1!}+\frac{y}{1!}\right)+\left(\frac{x^2}{2!}+\frac{xy}{1!1!}+\frac{y^2}{2!}\right)+\ldots$$
$$+\left\{\frac{x^r}{r!}+\frac{x^{r-1}y}{(r-1)!1!}+\frac{x^{r-2}y^2}{(r-2)!2!}+\ldots+\frac{y^r}{r!}\right\}+\ldots$$

But $$\frac{x^r}{r!}+\frac{x^{r-1}y}{(r-1)!1!}+\frac{x^{r-2}y^2}{(r-2)!2!}+\ldots+\frac{y^r}{r!}$$
$$=\frac{1}{r!}\left\{x^r+\frac{r!}{(r-1)!1!}x^{r-1}y\right.$$
$$\left.+\frac{r!}{(r-2)!2!}x^{r-2}y^2+\ldots+y^r\right\}$$
$$=\frac{1}{r!}\left\{x^r+{}^rC_1x^{r-1}y+{}^rC_2x^{r-2}y^2+\ldots+y^r\right\}$$
$$=(x+y)^r/r!$$

Therefore

$$f(x)\,.\,f(y)=1+\frac{x+y}{1!}+\frac{(x+y)^2}{2!}+\ldots+\frac{(x+y)^r}{r!}+\ldots,$$

i.e., $$f(x)\,.\,f(y)=f(x+y). \quad . \quad . \quad . \quad (1)$$

Again

$$f(x)\,.\,f(y)\,.\,f(z)=f(x+y)\,.\,f(z)=f(x+y+z),$$

and a repeated application of (1) gives

$$f(x) \,.\, f(y) \,.\, f(z) \ldots \text{ to } k \text{ factors}$$
$$= f(x+y+z+\ldots \text{to } k \text{ terms}). \quad (2)$$

Case I. Let x be a positive integer; then, by (2), $f(1) \,.\, f(1) \ldots$ to x factors $= f(1+1+\ldots$ to x terms$)$, i.e.,

$$\{f(1)\}^x = f(x).$$

But
$$f(1) = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots = e,$$

therefore
$$e^x = f(x).$$

Case II. Let x be a positive fraction equal to p/q, where p and q are positive integers; then

$$f(p/q) \,.\, f(p/q) \ldots \text{to } q \text{ factors} = f(q \,.\, p/q) = f(p).$$

But $f(p) = e^p$, by Case I, since p is a positive integer. Therefore

$$\{f(p/q)\}^q = e^p,$$

or, taking the qth root of both the sides,

$$f(p/q) = e^{p/q}.$$

Case III. Let x be negative and equal to $-y$, where y is a positive integer or fraction. Then, by (1),

$$f(-y) \,.\, f(y) = f(0) = 1,$$

so that
$$f(-y) = \frac{1}{f(y)} = \frac{1}{e^y} = e^{-y}.$$

Therefore for all values of x, $f(x) = e^x$, i.e.,

$$\mathbf{e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^r}{r!} + \ldots .}$$

This is known as the *Exponential Theorem.*

2·3. Irrationality of e. *To prove that e is irrational.*

If possible, suppose that $e=p/q$, where p and q are positive integers; then

$$\frac{p}{q}=\left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\dots+\frac{1}{q!}\right)+\frac{1}{(q+1)!}+\frac{1}{(q+2)!}+\dots.$$

Multiplying both the sides by $q!$ we get

$$p(q-1)!=(\text{an integer})+\frac{1}{q+1}+\frac{1}{(q+1)(q+2)}+\dots. \quad (1)$$

But $\dfrac{1}{q+1}+\dfrac{1}{(q+1)(q+2)}+\dfrac{1}{(q+1)(q+2)(q+3)}+\dots$ is a proper fraction, since it is less than the geometric series

$$\frac{1}{q+1}+\frac{1}{(q+1)^2}+\frac{1}{(q+1)^3}+\dots,$$

whose sum is $1/q$.

Thus equation (1) states that an integer is equal to an integer plus a proper fraction, which is impossible. Hence e cannot be expressed as a ratio of two integers, and is therefore irrational.

NOTE. Irrational numbers were formerly often called *incommensurable* numbers.

2·4. $\mathbf{e^x}$ as a limit. *To show that, for all values of x,*

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=e^x.$$

For sufficiently large values of n, x/n is less than unity. Therefore, by the Binomial Theorem,

$$\left(1+\frac{x}{n}\right)^n=1+n\frac{x}{n}+\frac{n(n-1)}{2!}\left(\frac{x}{n}\right)^2+\ldots$$
$$+\frac{n(n-1)\ldots(n-r+1)}{r!}\left(\frac{x}{n}\right)^r+\ldots$$
$$=1+x+\left(1-\frac{1}{n}\right)\frac{x^2}{2!}+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\frac{x^3}{3!}\ldots$$
$$+\left\{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{r-1}{n}\right)\right\}\frac{x^r}{r!}+\ldots.$$

Now, as n tends to infinity $1/n$, $2/n$, ... all tend to zero; therefore it appears that*

$$\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{r-1}{n}\right)\to 1.$$

Hence

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^r}{r!}+\ldots=e^x.$$

COROLLARY. Puting $x=1$, we see that

$$\mathbf{\lim_{n\to\infty}(1+1/n)^n=e.}$$

2·5. Summation of Series. The exponential theorem can be used to find the sum of certain types of series. For example, putting $-x$ for x in

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\frac{x^r}{r!}+\ldots, \tag{1}$$

we obtain

$$e^{-x}=1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots+(-1)^r\frac{x^r}{r!}+\ldots. \tag{2}$$

Therefore the sums of the series

$$1+\frac{x^2}{2!}+\frac{x^4}{4!}+\ldots \text{ and } x+\frac{x^3}{3!}+\frac{x^5}{5!}+\ldots, \tag{3}$$

*This st^tement really requires proof, but the proof is subtle and beyond our scope.

can be obtained by the addition and subtraction respectively of (1) and (2).

Any other series which can be thrown into one of the forms (1), (2) or (3), can be summed up. An example is given below.

Ex. Find the sum to infinity of the series

$$1+\frac{2^3}{2!}+\frac{3^3}{3!}+\frac{4^3}{4!}+\dots. \qquad [\textit{Agra}, 1955]$$

We have $r^3=r(r-1)(r-2)+3r^2-2r$

$$=r(r-1)(r-2)+3r(r-1)+r.$$

Therefore, if $r>2$,

$$\frac{r^3}{r!}=\frac{1}{(r-3)!}+\frac{3}{(r-2)!}+\frac{1}{(r-1)!}.$$

Also $$2^3/2!=3+1/1!$$

Hence the given series

$$=1+\left(3+\frac{1}{1!}\right)+\left(1+\frac{3}{1!}+\frac{1}{2!}\right)+\left(\frac{1}{1!}+\frac{3}{2!}+\frac{1}{3!}\right)$$
$$+\left(\frac{1}{2!}+\frac{3}{3!}+\frac{1}{4!}\right)+\dots$$

$$=\left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\dots\right)+3\left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\dots\right)$$
$$+\left(1+\frac{1}{1!}+\frac{1}{2!}+\dots\right)$$

$$=e+3e+e=5e.$$

EXAMPLES

1. Calculate $\sqrt{e}$ to four decimal places.
2. Expand $(e^{5x}+e^x)/e^{3x}$ in ascending powers of x.
3. Find the coefficient of x^n in the expansion of
$$(1-3x+x^2)/e^x.$$

Find the sum of the following series :

4. $1+\frac{2}{1!}+\frac{3}{2!}+\frac{4}{3!}+\ldots.$

5. $1+\frac{4^2}{3!}+\frac{4^4}{5!}+\ldots.$

6. $\frac{1^2}{2!}+\frac{2^2}{3!}+\frac{3^2}{4!}+\ldots.$ [*Delhi*, 1958]

7. $\frac{1}{2}-\frac{1}{3.1!}+\frac{1}{4.2!}-\frac{1}{5.3!}+\ldots.$

8. $1+\frac{2^3}{1!}x+\frac{3^3}{2!}x^2+\ldots+\frac{(n+1)^3}{n!}x^n+\ldots.$ [*Bombay*, 1948]

9. $\frac{2.3}{3!}+\frac{3.5}{4!}+\frac{4.7}{5!}+\frac{5.9}{6!}+\ldots.$ [*Mysore*, 1952]

10. $\sum_1^\infty \frac{1^2+2^2+3^2+\ldots+n^2}{n!}.$ [*Madras*, 1953]

11. Prove that

$$\frac{1^2.2^2}{1!}+\frac{2^2.3^2}{2!}+\frac{3^2.4^2}{3!}+\ldots\text{ad inf.}=27e.$$

[*Andhra*, 1953]

12. Prove that

$$1+\frac{1+3}{2!}+\frac{1+3+3^2}{3!}+\frac{1+3+3^2+3^3}{4!}+\ldots=\tfrac{1}{2}e(e^2-1)$$

[*Madras*, 1953]

2·6. Logarithms.

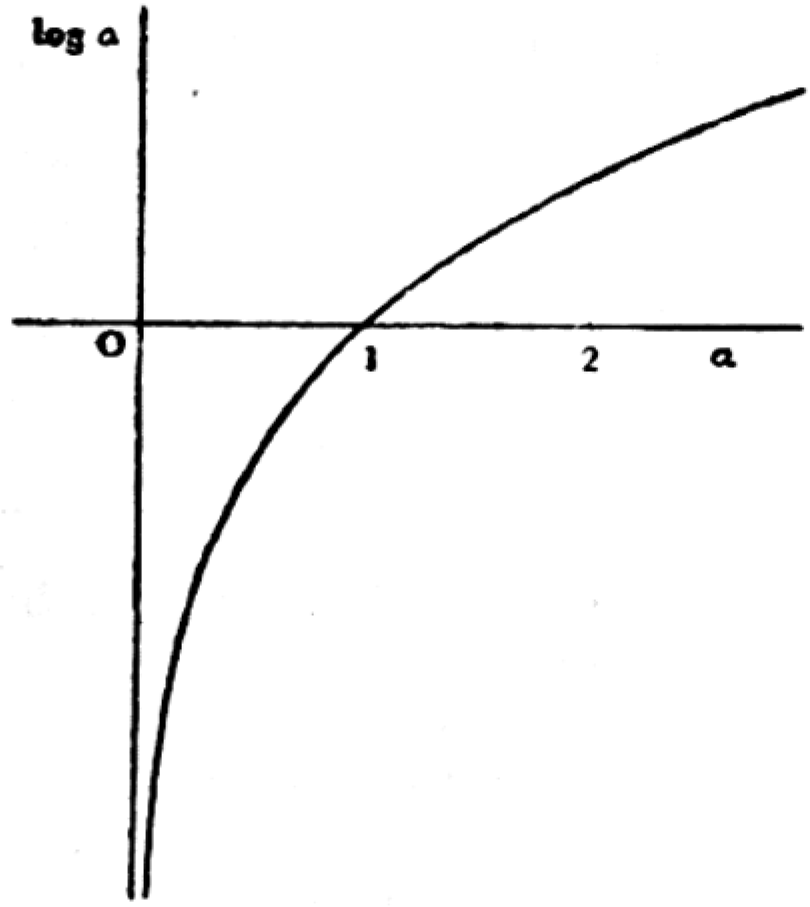
If $e^x=a$, then x is called the *logarithm* of a, and we write

$$x=\log a. \qquad \ldots \quad (1)$$

Thus from the definition of logarithms we see that

$$e^{\log a}=a, \quad \text{and} \quad \log e^x=x.$$

Since e is greater than unity, e^x is greater than unity if x is positive, and less than unity if x is negative. Also, $e^0=1$. Therefore, by (1), $\log a$ is positive if $a>1$, negative if $a<1$, and zero if $a=1$.



The graph of $\log a$ for various values of a is shown in the margin. It is seen that the curve does not extend to negative values of a, the reason being that e^x is always positive for real values of x.

We can give a more general definition of the logarithm. If $t^x=a$, then we say that x is the logarithm of a to the base t; and we write $x=\log_t a$.

From the theory of indices we can prove that

$$\log ab=\log a+\log b,\ \log(a/b)=\log a-\log b,$$

and

$$\log a^m=m\log a,$$

to whatever base the logarithms are taken. These formulae are of great help in numerical computation.

The logarithm to the base e is known as the *natural* logarithm, or the Napierian logarithm, after Napier, the inventor of logarithms. In theoretical work we mostly employ natural logarithms, and so the suffix e is generally omitted, the base e being understood.

The logarithms to the base 10 are known as *common* logarithms. In numerical work common logarithms are invariably employed on account of convenience, and the base is understood to be 10 when not mentioned. Thus log 2 generally means $\log_{10} 2$, while $\log a$ means $\log_e a$. In case there is a likelihood of confusion, the base should be mentioned.

2·7. Expansion of a^x. *To expand a^x in ascending powers of x.*

We have by the exponential theorem

$$e^{cx} = 1 + cx + \frac{(cx)^2}{2!} + \frac{(cx)^3}{3!} + \dots .$$

Put $e^c = a$, so that $e^{cx} = a^x$ and $c = \log_e a$; then

$$a^x = 1 + x \log_e a + \frac{(x \log_e a)^2}{2!} + \frac{(x \log_e a)^3}{3!} + \dots .$$

2·8. Logarithmic Series. *To expand $\log_e(1+x)$ in ascending powers of x.*

In the expansion

$$a^y = 1 + y \log_e a + \frac{(y \log_e a)^2}{2!} + \frac{(y \log_e a)^3}{3!} + \dots ,$$

put $(1+x)$ for a; then

$$(1+x)^y = 1 + y \log_e(1+x) + \{y \log_e(1+x)\}^2/2! + \dots (1)$$

If x is numerically less than unity, we also have

$$(1+x)^y = 1 + yx + \frac{y(y-1)}{2!}x^2 + \frac{y(y-1)(y-2)x^3}{3!} + \dots , \quad . \quad . \quad . \quad (2)$$

by the Binomial Theorem. If this expansion be arranged in powers of y, the coefficient of y will be

$$x + \frac{(-1)}{2!}x^2 + \frac{(-1)(-2)}{3!}x^3 + \frac{(-1)(-2)(-3)}{4!}x^4 + \dots ,$$

that is, $\quad x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots .$

But (1) and (2) are different expansions of the same expression. Therefore the coefficients of y in the two must be equal. Hence

$$\mathbf{\log_e(1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \dots .}$$

the expansion being valid when x is numerically less than unity.

This is known as the logarithmic series.

Ex. If α and β be the roots of $x^2-px+q=0$, show that $\log(1+px+qx^2)=(\alpha+\beta)x-\frac{1}{2}(\alpha^2+\beta^2)x^2+\frac{1}{3}(\alpha^3+\beta^3)x^3-\ldots$. [*Allahabad*, 1950]

Since α and β are the roots of $x^2-px+q=0$, therefore

$$\alpha+\beta=p \text{ and } \alpha\beta=q.$$

Therefore $\log(1+px+qx^2)=\log\{1+(\alpha+\beta)x+\alpha\beta x^2\}$

$$=\log\{(1+\alpha x)(1+\beta x)\}=\log(1+\alpha x)+\log(1+\beta x)$$

$$=\alpha x-\tfrac{1}{2}\alpha^2x^2+\tfrac{1}{3}\alpha^3x^3-\ldots+\beta x-\tfrac{1}{2}\beta^2x^2+\tfrac{1}{3}\beta^3x^3-\ldots$$

$$=(\alpha+\beta)x-\tfrac{1}{2}(\alpha^2+\beta^2)x^2+\tfrac{1}{3}(\alpha^3+\beta^3)x^3-\ldots.$$

2·9. Calculation of Logarithms. The series

$$\log_e(1+x)=x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\ldots \qquad (1)$$

can be used to calculate $\log(1+x)$ only when x is less than unity; and even then it is not very convenient to use unless x is small, since the terms decrease slowly and a large number of terms are required to give sufficient accuracy. We give below two series which are more convenient in calculating logarithms.

Putting $-x$ for x in (1), we get

$$\log_e(1-x)=-x-\tfrac{1}{2}x^2-\tfrac{1}{3}x^3-\tfrac{1}{4}x^4-\ldots. \qquad (2)$$

Subtracting (2) from (1),

$$\log_e\frac{1+x}{1-x}=2(x+\tfrac{1}{3}x^3+\tfrac{1}{5}x^5+\ldots). \qquad (3)$$

If we put $x=1/n$, this gives

$$\log_e(n+1)-\log_e(n-1)=2\left(\frac{1}{n}+\frac{1}{3n^3}+\frac{1}{5n^5}+\ldots\right); \qquad (4)$$

and if we put

$$\frac{1+x}{1-x}=\frac{n+1}{n}, \text{ i.e., } x=\frac{1}{2n+1},$$

(3) gives

$$\log_e(n+1)-\log_e n$$
$$=2\left\{\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\dots\right\}. \quad (5)$$

Both (4) and (5) are convenient for calculating logarithms, n being so chosen that we know either $\log(n+1)$, $\log n$ or $\log(n-1)$. Two more formulae can be obtained by putting $x=1/n$ in (1) and (2). But in these the terms do not decrease as rapidly as in (4) or (5).

The above series give Napierian logarithms. To obtain the common logarithms (i.e., logarithms to the base 10), Napierian logarithms have to be multiplied by $1/\log_e 10$, which is called the *modulus* of the common system. The reason for this is given below.

If $\log_{10} n=x$, then $n=10^x$. Therefore

$$\log_e n=\log_e 10^x=x\log_e 10,$$

so that

$$x=\log_e n/\log_e 10,$$

that is,

$$\log_{10} n=\log_e n/\log_e 10.$$

Ex. Find $\log_{10} 2$ to four decimal places.

Putting $n=3$ in (4) we get

$$\log_e 4-\log_e 2=2\left(\frac{1}{3}+\frac{1}{3.3^3}+\frac{1}{5.3^5}+\frac{1}{7.3^7}+\dots\right)$$
$$=2(0{\cdot}33333+0{\cdot}01235+0{\cdot}00082+0{\cdot}00007+\dots)=0{\cdot}69314,$$

i.e.,

$$\log_e 2=0{\cdot}69314.$$

Putting $n=9$ in (4) we get

$$\log_e 10-\log_e 8=2\left(\frac{1}{9}+\frac{1}{3.9^3}+\frac{1}{5.9^5}+\dots\right)$$

or $\quad \log_e 10-3\log_e 2=2(0{\cdot}11111+0{\cdot}00046+\dots)=0{\cdot}22314.$

Therefore $\quad \log_e 10=3\log_e 2+0{\cdot}22314=2{\cdot}30256.$

Hence $\quad \log_{10} 2=0{\cdot}69314/2{\cdot}30256=0{\cdot}3010.$

Examples

Expand the following expressions, giving the general term.

1. $\log(2+x)$.
2. $\log(1+3x+2x^2)$.

3. $\log(1-x)^{(1-x)}$.

4. $\log\{1/(1-x-x^2+x^3)\}$. [*Madras*, 1948]

5. Show that the coefficient of x^n in the expansion of $\log_e(1+x+x^2)$ in ascending powers of x is $1/n$ or $-2/n$ according as n is not or is a multiple of 3. [*Bombay*, 1948]

6. Show that $\log_e 10 = 3\log_e 2 + \frac{1}{4} - \frac{1}{2}(\frac{1}{4})^2 + \frac{1}{3}(\frac{1}{4})^3 - \ldots$. [*Madras*, 1949]

7. Find the value of

$$2\left(\frac{1}{7}+\frac{1}{3}\cdot\frac{1}{7^3}+\frac{1}{5}\cdot\frac{1}{7^5}+\ldots\right).$$ [*Annamalai*, 1949]

8. Show that, if x is numerically less than unity,

$$\tfrac{1}{2}x^2+\tfrac{2}{3}x^3+\tfrac{3}{4}x^4+\tfrac{4}{5}x^5+\ldots=x/(1-x)+\log(1-x).$$

9. If $y=x/(x+1)$ and $0<x<1$, express $x-\frac{1}{2}x^2+\frac{1}{3}x^3-\ldots$ as a series in powers of y.

10. Find the common logarithms of 7, 11 and 13; given that $\log_{10}2=0{\cdot}30103$ and $\log_{10}e=0{\cdot}43429$.

11. Show that

$$\frac{x-1}{x+1}+\frac{x^2-1}{2(x+1)^2}+\frac{x^3-1}{3(x+1)^3}+\ldots=\log x.$$ [*Madras*, 1954]

12. Find the sum of the following infinite series

(i) $\log_3 e-\log_9 e+\log_{27} e-\log_{81} e+\ldots$. [*Madras*, 1958]

(ii) $\frac{1}{3}x^2+\frac{1}{15}x^4+\ldots+x^{2n}/(4n^2-1)+\ldots$, $|x|\leqslant 1$. [*Andhra*, 1952]

EXAMPLES ON CHAPTER II

1. Find the coefficient of x in the infinite series

$$1+\frac{(a+bx)}{1!}+\frac{(a+bx)^2}{2!}+\frac{(a+bx)^3}{3!}+\ldots.$$ [*U.P.C.S.*, 1947]

2. Prove that

$$\left(1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\ldots\right)^2=1+\left(1+\frac{1}{3!}+\frac{1}{5!}+\ldots\right)^2.$$

[*Madras*, 1949]

3. If $y=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\ldots$, show that

$$x=y+\frac{y^2}{2!}+\frac{y^3}{3!}+\frac{y^4}{4!}+\ldots.$$ [*Gujarat*, '60]

4. Prove that

$$\log\{(1+2e^x)/3\}=\tfrac{2}{3}x+\tfrac{1}{9}x^2-\tfrac{1}{81}x^3,$$

higher powers of x being neglected. [*Andhra*, 1950]

5. Show that

$$\log_e 3=1+\frac{1}{3.2^2}+\frac{1}{5.2^4}+\frac{1}{7.2^6}+\ldots.$$ [*Patna*, 1949]

6. Show that if p and q are positive

$$\log_e\frac{p}{q}=2\left\{\left(\frac{p-q}{p+q}\right)+\tfrac{1}{3}\left(\frac{p-q}{p+q}\right)^3+\tfrac{1}{5}\left(\frac{p-q}{p+q}\right)^5+\cdots\right\}.$$

[*Rangoon*, 1950]

7. Find the Napierian logarithm of 1001/999 correct to seven places of decimals. [*Allahabad*, 1950]

8. If $x\neq 1$, show that

$$\tfrac{1}{2}\log\left(\frac{1+x}{1-x}\right)^2=\frac{2x}{1+x^2}+\tfrac{1}{3}\left(\frac{2x}{1+x^2}\right)^3+\tfrac{1}{5}\left(\frac{2x}{1+x^2}\right)^5+\ldots.$$

[*Mysore*, 1949]

9. Prove that

$$\log\{(1+x)^{1+x}(1-x)^{1-x}\}=2\left(\frac{x^2}{1.2}+\frac{x^4}{3.4}+\frac{x^6}{5.6}+\ldots\right).$$

10. If $x>1$, prove that

$$2\log_e x-\log_e(x+1)-\log_e(x-1)=\frac{1}{x^2}+\frac{1}{2x^4}+\frac{1}{3x^6}+\ldots.$$

[*Agra*, 1941]

11. Show that

$$2\log_e n-\log_e(n+1)-\log_e(n-1)$$
$$=2\left\{\frac{1}{2n^2-1}+\frac{1}{3(2n^2-1)^3}+\frac{1}{5(2n^2-1)^5}+\ldots\right\}.$$

12. Show that

$$\frac{1}{n+1}+\frac{1}{2(n+1)^2}+\frac{1}{3(n+1)^3}+\ldots=\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\ldots.$$

[*Agra*, 1953]

13. Show that

$$\log\left(1+\frac{1}{n}\right)^n=1-\frac{1}{2(n+1)}-\frac{1}{2\,.\,3(n+1)^2}-\frac{1}{3\,.\,4(n+1)^3}-\ldots.$$

[*Rajasthan*, 1960]

14. Show that

$$n+\frac{1}{n}=2\left\{1+\frac{(\log_e n)^4}{2!}+\frac{(\log_e n)^2}{4!}+\ldots\right\}.$$ [*Patna*, '50]

15. Find the sum of

$$1+\frac{\log_e 2}{2!}+\frac{(\log_e 2)^2}{3!}+\ldots.$$ [*Bombay*, 1947]

16. Find the sum to infinity of the series

$$1+\frac{1+2}{2!}+\frac{1+2+3}{3!}+\frac{1+2+3+4}{4!}+\ldots.$$ [*Poona*, '60]

17. Show that

$$1+\frac{1+2}{2!}+\frac{1+2+2^2}{3!}+\frac{1+2+2^2+2^3}{4!}+\ldots \text{ to } \infty=e^2-e.$$

[*Rangoon*, 1950]

18. Prove that $e^{-1}=2\left(\frac{1}{3!}+\frac{2}{5!}+\frac{3}{7!}+\ldots\right)$. [*Osmania*, '52]

19. Show that

$$\log\frac{4}{e}=\frac{1}{1\,.\,2}-\frac{1}{2\,.\,3}+\frac{1}{3\,.\,4}-\frac{1}{4\,.\,5}+\ldots.$$ [*Andhra*, 1950]

20. If α and β are the roots of the equation $x^2-px+q=0$, show that

$$-\log_e(1-px+qx^2)=(\alpha+\beta)x+\tfrac{1}{2}(\alpha^2+\beta^2)x^2+\ldots$$
$$+\{(\alpha^n+\beta^n)/n\}x^n+\ldots.$$ [*Utkal*, 1950]

21. If $\log(1+x+x^2+x^3)$ be expanded in ascending powers of x, show that the coefficient of x^n is $1/n$ if n is odd or of the form $4m+2$, and $-3/n$ if n is of the form $4m$.

22. Expand $\log\{(1+x+x^2)/(1-x+x^2)\}$ in ascending powers of x. [*Bombay*, 1954]

23. If $\log(1-x+x^2)$ be expanded in ascending powers of x in the form $a_1x+a_2x^2+a_3x^3+\ldots$, show that

$$a_3+a_6+a_9+\ldots=\tfrac{2}{3}\log_e 2.$$ [*Rajputana*, 1950]

24. Apply the exponential and binomial theorems to show that

$$n^n-n(n-1)^n+\frac{n(n-1)}{2!}(n-2)^n+\ldots=n!$$ [*Poona*, 1952]

CHAPTER III

INEQUALITIES

3·1. Definitions. A number a is said to be greater than another number b if $a-b$ is positive, and a is said to be less than b if $a-b$ is negative.

Thus 1 is greater than -2 because $1-(-2)$ is positive; and -3 is less than -2 because $-3-(-2)$ is negative. For brevity we use the sign $>$ for "is greater than" and $<$ for "is less than", and write $1>-2$ and $-3<-2$.

In the present chapter we shall suppose (unless stated otherwise) that the letters denote positive real numbers. Some authors use the word "quantity" instead of "number", but in algebra we really deal with numbers.

3·2. Manipulation of Inequalities. The rules for the manipulation of inequalities are mostly the same as for equations.

(i) Thus, if $a>b$, it follows from the definition given above that

$$a+c>b+c, \quad . \quad . \quad . \quad (1)$$

$$a-c>b-c,$$

$$ac>bc, \quad . \quad . \quad . \quad (2)$$

$$a/c>b/c;$$

that is, *an inequality will still hold if both its sides are increased, diminished, multiplied, or divided by the same positive number.*

(ii) If $$a-c>b,$$
then, adding c to each side, we get
$$a>b+c;$$
which shows that *a term on one side of an inequality may be transposed to the other side after changing its sign.*

(iii) If $a>b$, $a-b$ is positive and $-c(a-b)$ is negative. Therefore
$$-ac<-bc,$$
i.e., *if the sides of an inequality be multiplied by a negative number, the sign of inequality is reversed.*

In particular, if $a>b$,
$$-a<-b.$$

(iv) If $a_1>b_1$, $a_2>b_2$, $a_3>b_3, \dots, a_n>b_n$, it follows from the definition of an inequality that
$$a_1+a_2+a_3+\dots+a_n>b_1+b_2+b_3+\dots+b_n.$$
Also, successive applications of (2) give
$$a_1a_2a_3\dots a_n>b_1a_2a_3\dots a_n>b_1b_2a_3\dots a_n>\dots>b_1b_2b_3\dots b_n;$$
or, taking the first and the last members, we get
$$a_1a_2a_3\dots a_n>b_1b_2b_3\dots b_n. \quad . \quad . \quad . \quad . \quad (3)$$

(v) From (3) we deduce that, if $a>b$ and n is a positive integer,
$$a^n>b^n. \quad . \quad . \quad . \quad (4)$$
It can be shown that this result holds even when n is fractional, provided we take the positive and real values of a^n and b^n.

Dividing both sides of (4) by a^nb^n, we see that, if $a>b$,
$$a^{-n}<b^{-n}.$$

(vi) From the properties of the exponential and logarithmic functions (see § 2·6) it follows that, if $a>b$,

$$e^a>e^b,$$

and $$\log a>\log b.$$

3·21. Factorisation. For those inequalities in which the expression obtained by taking the difference of the two sides can be factorised, we can prove the inequality by considering the sign of each separate factor. Sometimes a special selection of factors is useful.

When both sides of an inequality are symmetrical in a and b, there is no loss of generality in assuming that $a>b$. For, if $b>a$, we can write a for b and b for a, and, because of the symmetry, the inequality will remain unaltered. Similarly, if an inequality is symmetrical in $a, b, c, \ldots,$ we can assume that $a>b>c>\ldots$.

Ex. 1. Prove that $a^3b+ab^3<a^4+b^4$.

We have
$$\begin{aligned} a^3b+ab^3-a^4-b^4 &= a^3(b-a)+b^3(a-b) \\ &= (a-b)(b^3-a^3) \\ &= -(a-b)^2(a^2+ab+b^2), \end{aligned}$$
which is negative. Therefore
$$a^3b+ab^3<a^4+b^4.$$

Ex. 2. Show that $(n!)^2>n^n$. [*Baroda*, 1960]

Rearranging the factors in $(n!)^2$, we see that

$$(n!)^2=\{1\,.\,n\}\{2(n-1)\}\{3(n-2)\}\ldots\{r(n-r+1)\}\ldots\{n\,.\,1\}. \quad (1)$$

Now $$r(n-r+1)>n \quad . \quad . \quad . \quad (2)$$

if $$r(n-r+1)-n>0,$$

i.e., if $$(n-r)(r-1)>0.$$

But this condition is satisfied when r is less than n and greater than 1, i.e. for $r=2, 3, \ldots, (n-1)$. For these values of r, (2) becomes

$$2(n-1)>n, 3(n-2)>n, \ldots, (n-1)2>n.$$

By multiplication we get

$$\{2(n-1)\}\{(3(n-2)\}\ldots\{(n-1)2\}>n^{n-2}. \qquad (3)$$

Multiplying both sides by $\{1 \, . \, n\}^2$, we get

$$\{1 \, . \, n\}\{2(n-1)\}\{3(n-2)\}\ldots\{(n-1)2\}\{n \, . \, 1\}>n^n,$$

or, by (1), $$(n!)^2>n^n.$$

Ex. 3. Show that the expression

$$a(a-b)(a-c)+b(b-c)(b-a)+c(c-a)(c-b)$$

is positive.

Take $a>b>c$. Then from the inequality $a>b$, we get

$$a-c>b-c,$$

and $$a(a-b)>b(a-b).$$

Multiplying together the respective sides, we get

$$a(a-b)(a-c)>b(b-c)(a-b),$$

or, by transposition,

$$a(a-b)(a-c)+b(b-c)(b-a)>0. \quad . \quad . \quad (1)$$

Also, since $a>b>c$, therefore $c-a$ and $c-b$ are both negative. Hence

$$c(c-a)(c-b)>0. \quad . \quad . \quad . \quad (2)$$

Adding (1) and (2), we get the required result.

EXAMPLES

1. If $a>b$ and $x>0$, prove that $(a+x)/(b+x)<a/b$.
2. If $a>b$, show that $a^a b^b>a^b b^a$.
3. Find which is the greater, $3ab^2$ or a^3+2b^3.
4. Prove that $a^3+b^3>a^2b+ab^2$.
5. If x may have any real value, find which is the greater, x^3-1 or x^2-x.
6. Show that $x^3+13a^2x>5ax^2+9a^3$ if $x>a$.

7. Find the greatest value of x for which $6x^2+10$ is greater than x^3+13x.

Prove the following inequalities :

8. $2(ab+1)>(a+1)(b+1)$, if $a>1$, $b>1$.

9. $(a^4+b^4)(a^5+b^5)<2(a^9+b^9)$.

10. $a^2(a-b)(a-c)+b^2(b-c)(b-a)+c^2(c-a)(c-b)>0$.

11. $(n!)^2<r!(2n-r)!$

12. If a, b, c, d are in harmonical progression, prove that

$$a+d>b+c.$$

[Hint. Take $p-3q$, $p-q$, $p+q$, $p+3q$ as the reciprocals of a, b, c, d.]

3·3. Arithmetic and Geometric means. The square of every real number is positive. So, if $x\neq y$, $(x-y)^2$ is positive, i.e.,

$$x^2-2xy+y^2>0,$$

or

$$x^2+y^2>2xy.$$

Putting a and b for x^2 and y^2, we see that if a and b are positive and unequal,

$$\tfrac{1}{2}(\mathbf{a}+\mathbf{b})>\sqrt{(\mathbf{ab})},$$

i.e., *the arithmetic mean of two unequal positive numbers is greater than their geometric mean.*

Ex. Show that $a^2+b^2+c^2>bc+ca+ab$.

We have $\quad \tfrac{1}{2}(b^2+c^2)>bc,$

$$\tfrac{1}{2}(c^2+a^2)>ca,$$

and $\quad \tfrac{1}{2}(a^2+b^2)>ab;$

whence, by addition, we get the result.

3·31. Sum and Product of two numbers. Let a and b be two positive numbers, S their sum and P their product. Then from the identity

$$(a+b)^2-(a-b)^2=4ab,$$

we get

$$4P=S^2-(a-b)^2. \quad . \quad . \quad . \quad (1)$$

Thus if a and b vary, subject to the condition that their sum S remains constant, their product P will be greatest when $a=b$.

Also, since (1) may be written as

$$S^2=4P+(a-b)^2,$$

if a and b vary, P remaining constant, S will be least when $a=b$.

In other words, *if the sum of two positive numbers be given, their product is greatest when they are equal; and if their product be given, their sum is least when they are equal.*

3·4. Greatest Value of a Product. *To find the greatest value of a product the sum of whose factors is constant.*

Let there be n factors $a, b, c, \ldots, k$; and denote their constant sum $a+b+c+\ldots+k$ by s.

Consider the product $abc\ldots k$. If a and b are any two unequal factors, and we replace them by the two equal factors $\frac{1}{2}(a+b)$, $\frac{1}{2}(a+b)$, the product is increased, while the sum remains unaltered.

Therefore, as long as any two of the factors $a, b, c, \ldots, k$, are unequal, we can increase their product without altering their sum. Hence the product is greatest when all the factors are equal. In this case each of the n factors is equal to s/n, and thus the greatest value of the product is $(s/n)^n$, i.e.,

$$\left(\frac{a+b+c+\ldots+k}{n}\right)^n.$$

3·41. Definitions. If $a, b, c, \ldots, k$ are n numbers, then

$$\frac{a+b+c+\ldots+k}{n}$$

is called their *arithmetic mean*, and

$$\sqrt[n]{(abc\ldots k)}$$

is called their *geometric mean.*

3·42. Arithmetic and Geometric Means. *The arithmetic mean of n positive numbers which are not all equal to one another, is greater than their geometric mean.*

Let $a, b, c, \ldots, k$ be the n numbers, not all equal to one another. Their product is $abc\ldots k$, and their sum $a+b+c+\ldots+k$.

Now we know that the product of n factors, whose sum is given, is greatest when the factors are all equal to one another. Therefore

$$\{(a+b+c+\ldots+k)/n\}^n$$

is the greatest product of n factors having the sum $a+b+c+\ldots+k$, and consequently is greater than $abc\ldots k$.

Taking the nth root, it follows that

$$\frac{\mathbf{a+b+c+\ldots+k}}{\mathbf{n}} > (\mathbf{abc\ldots k})^{1/n},$$

which proves the proposition.

We give below an independent proof of this important theorem.

Case I. *n is a power of* 2. Let the n numbers be $a, b, c, d, \ldots, k$. Then

$$\tfrac{1}{2}(a+b) \geqslant (ab)^{1/2}, \ \tfrac{1}{2}(c+d) \geqslant (cd)^{1/2}, \ldots, \quad \ldots \quad (1)$$

the sign of equality being taken when the two numbers are equal. From (1) we get

$$\tfrac{1}{2}\{\tfrac{1}{2}(a+b)+\tfrac{1}{2}(c+d)\}\geqslant\{(ab)^{1/2}(cd)^{1/2}\}^{1/2}, \ldots,$$

i.e.,
$$\tfrac{1}{4}(a+b+c+d)\geqslant(abcd)^{1/4}, \ldots .$$

Proceeding in this way, we can show that

$$\frac{a+b+c+\ldots+k}{n}\geqslant(abc\ldots k)^{1/n}, \qquad . \quad . \quad (2)$$

when n is a power of 2.

Case II. *n is not a power of* 2. Take an integer r such that $n+r$ is a power of 2, and consider the numbers

$$a, b, c, \ldots, k, A, A, \ldots, A,$$

where A, the arithmetic mean of the numbers $a, b, c, \ldots, k$, occurs r times. Then, applying the result of Case I to these numbers, we get

$$\frac{a+b+c+\ldots+k+rA}{n+r}\geqslant(abc\ldots kA^r)^{1/(n+r)}.$$

Now, since $a+b+c+\ldots+k=nA$, the left-hand side is equal to A. Therefore

$$A^{n+r}\geqslant abc\ldots kA^r.$$

Cancelling the common factor A^r and taking the nth root of both the sides, this gives

$$A\geqslant\sqrt[n]{(abc\ldots k)},$$

the sign of equality being taken only when the numbers are all equal.

Ex. Show that $(n!)^3<n^n\{\tfrac{1}{2}(n+1)\}^{2n}$.

We have $(1^3 . 2^3 . 3^3 \ldots n^3)^{1/n}<(1^3+2^3+3^3+\ldots+n^3)/n$,

or
$$(n!)^{3/n}<\tfrac{1}{4}n(n+1)^2.$$

Raising both sides to the power n, we get the result.

3·43. Repeated Factors. *To find the greatest value of $a^m b^n c^p \ldots$ when $a+b+c+\ldots$ is constant; m, n, p, ... being given positive integers.*

Since $m, n, p, \ldots$ are constants, $a^m b^n c^p \ldots$ will be greatest when

$$\left(\frac{a}{m}\right)^m\left(\frac{b}{n}\right)^n\left(\frac{c}{p}\right)^p\ldots \qquad . \quad . \quad . \quad (1)$$

is greatest. This consists of m factors each equal to a/m, n factors each equal to b/n, p factors each equal to c/p, and so on. The sum of these factors is equal to

$$m(a/m)+n(b/n)+p(c/n)+\ldots, \text{ i.e., } a+b+c+\ldots,$$

which is a constant. Therefore (1) will be greatest when all the factors are equal to each other, i.e., when

$$\frac{a}{m}=\frac{b}{n}=\frac{c}{p}=\ldots=\frac{a+b+c+\ldots}{m+n+p+\ldots}.$$

Thus the greatest value of $a^m b^n c^p \ldots$ is

$$m^m n^n p^p \ldots \left(\frac{a+b+c+\ldots}{m+n+p+\ldots}\right)^{m+n+p+\ldots}.$$

3·44. Maxima and Minima. The foregoing articles can be used to solve questions on maxima and minima. But in the case of quadratic functions, and also in some other cases, it is more convenient to express the given function as the sum of a constant and the square of a variable expression. This method is illustrated in Ex. 3 below.

Ex. 1. Find the greatest value of $(b+x)^3(a-x)^2$ when x lies between a and $-b$.

Since the sum of $b+x$ and $a-x$ is constant, $(b+x)^3(a-x)^2$ will be greatest when

$$\frac{b+x}{3}=\frac{a-x}{2}=\frac{a+b}{5},$$

i.e., when $x=\frac{1}{5}(3a-2b)$; and the greatest value will be

$$3^3 \,.\, 2^2\{(a+b)/5\}^5.$$

Ex. 2. If $2x+5y=3$, find the greatest value of x^3y^4.

x^3y^4 is greatest when $(\frac{2}{3}x)^3(\frac{5}{4}y)^4$ is greatest. But the sum of the factors in the latter product is $3(\frac{2}{3}x)+4(\frac{5}{4}y)$, which is equal to 3, a constant. Therefore $(\frac{2}{3}x)^3(\frac{5}{4}y)^4$ is greatest when its factors are equal, i.e., when

$$\frac{2x}{3}=\frac{5y}{4}=\frac{2x+5y}{3+4}=\frac{3}{7};$$

and so the greatest value of x^3y^4 is

$$(\tfrac{9}{14})^3(\tfrac{12}{35})^4=2^5.3^{10}/5^4.7^7.$$

Ex. 3. Find the minimum value of $(x+a)(x+b)/(x-c)$.

Put $x-c=y$; then the given expression

$$=(y+c+a)(y+c+b)/y$$

$$=y+\{(c+a)(c+b)/y\}+c+a+c+b$$

$$=\left[\frac{\sqrt{\{(c+a)(c+b)\}}}{\sqrt{y}}-\sqrt{y}\right]^2+c+a+c+b$$

$$+2\sqrt{\{(c+a)(c+b)\}}. \quad (1)$$

This will be least when the squared term is zero, i.e., when

$$y=\sqrt{\{(c+a)(c+b)\}},$$

or
$$x=c+\sqrt{\{(c+a)(c+b)\}};$$

and the minimum value of (1) is

$$c+a+c+b+2\sqrt{\{(c+a)(c+b)\}}=\{\sqrt{(c+a)}+\sqrt{(c+b)}\}^2.$$

Examples

Prove that

1. $(ab+xy)(ax+by)>4abxy$.
2. $(b+c)(c+a)(a+b)>8abc$.
3. $b^2c^2+c^2a^2+a^2b^2>abc(a+b+c)$.
4. $6abc<bc(b+c)+ca(c+a)+ab(a+b)$.
5. $cd(a+b)^2<(ad+bc)(ac+bd)$.
6. Show that the sum of any positive number and its reciprocal is never less than 2.
7. If $a^2+b^2+c^2=1$ and $x^2+y^2+z^2=1$, show that
$$ax+by+cz<1.$$

Prove that

8. $a^2b+b^2c+c^2a>3abc$. [*Banaras*, 1955]
9. $(a+b+c)(bc+ca+ab)>9abc$. [*Baroda*, 1960]

10. $(x^2y+y^2z+z^2x)(xy^2+yz^2+zx^2)>9x^2y^2z^2$.

11. $\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{d}{a}>4$. [*Baroda*, 1960]

12. $\left(\frac{a}{e}+\frac{b}{f}+\frac{c}{g}\right)\left(\frac{e}{a}+\frac{f}{b}+\frac{g}{c}\right)>9$.

13. $n^n>1.3.5\ldots(2n-1)$. [*Gorakhpur*, 1960]

14. $2.4.6\ldots2n<(n+1)^n$. [*Kashmir*, 1954]

15. $(1^r+2^r+3^r+\ldots+n^r)^n>n^n(n!)^r$. [*Panjab*, 1951]

16. If $x>1$ and n is a positive integer, prove that

$$\frac{x^n-1}{x-1}>nx^{(n-1)/2}.$$

17. Find the minimum value of $x^2-10x+27$, and the maximum value of $16x-13-4x^2$.

18. Given that x lies between -6 and 8, find the maximum value of $(8-x)^3(x+6)^4$. [*Annamalai*, 1950]

19. Find the greatest value of x^2y^3 if $3x+2y=1$.

20. Find the minimum value of $(5+x)(2+x)/(1+x)$.

3·5. Mean of the mth powers. *If a and b are positive and unequal,*

$$\frac{a^m+b^m}{2}>\left(\frac{a+b}{2}\right)^m,$$

except when m lies between 0 and 1.

We can write

$$a^m=\left(\frac{a+b}{2}+\frac{a-b}{2}\right)^m=\left(\frac{a+b}{2}\right)^m\left(1+\frac{a-b}{a+b}\right)$$

$$=\left(\frac{a+b}{2}\right)\left\{1+m\left(\frac{a-b}{a+b}\right)+\frac{m(m-1)}{2!}\left(\frac{a-b}{a+b}\right)^2\right.$$

$$\left.+\frac{m(m-1)(m-2)}{3!}\left(\frac{a-b}{a+b}\right)^3+\ldots\right\},$$

on expansion by the Binomial Theorem.

Similarly,

$$b^m=\left(\frac{a+b}{2}-\frac{a-b}{2}\right)^m=\left(\frac{a+b}{2}\right)^m\left(1-\frac{a-b}{a+b}\right)^m$$

$$=\left(\frac{a+b}{2}\right)^m\left\{1-m\left(\frac{a-b}{a+b}\right)+\frac{m(m-1)}{2!}\left(\frac{a-b}{a+b}\right)^2\right.$$

$$\left.-\frac{m(m-1)(m-2)}{3!}\left(\frac{a-b}{a+b}\right)^3+\dots\right\}.$$

On adding these two expansions the alternate terms cancel, and we obtain

$$\frac{a^m+b^m}{2}=\left(\frac{a+b}{2}\right)^m+\frac{m(m-1)}{2!}\left(\frac{a+b}{2}\right)^{m-2}\left(\frac{a-b}{2}\right)^2$$

$$+\frac{m(m-1)(m-2)(m-3)}{4!}\left(\frac{a+b}{2}\right)^{m-4}\left(\frac{a-b}{2}\right)^4+\dots.$$

Now three cases arise.

(i) If $m<0$, i.e., if m is negative, all the terms on the right-hand side are positive. Therefore

$$\frac{a^m+b^m}{2}>\left(\frac{a+b}{2}\right)^m.$$

(ii) If m lies between 0 and 1, all the terms on the right after the first are negative. Therefore

$$\frac{a^m+b^m}{2}<\left(\frac{a+b}{2}\right)^m.$$

(iii) If $m>1$, put $m=1/n$, where $n<1$. Then, if A and B are any two positi e numbers, we get by (ii)

$$\left(\frac{A+B}{2}\right)^n>\frac{A^n+B^n}{2},$$

whence

$$\frac{A+B}{2}>\left(\frac{A^n+B^n}{2}\right)^{1/n}.$$

Now take $A=a^m$, $B=b^m$, then

$$\frac{a^m+b^m}{2}>\left(\frac{a^{mn}+b^{mn}}{2}\right)^{1/n},$$

i.e.,
$$\frac{\mathbf{a}^m+\mathbf{b}^m}{2}>\left(\frac{\mathbf{a}+\mathbf{b}}{2}\right)^m.$$

Hence the theorem is established. If $m=0$ or 1, the inequality becomes an equality.

3·51. Mean of mth powers. General Case. *If $a, b, c, \ldots, k$ are n positive numbers,*

$$\frac{a^m+b^m+c^m+\ldots+k^m}{n}>\left(\frac{a+b+c+\ldots+k}{n}\right)^m,$$

except when m lies between 0 *and* 1.

Suppose that m does not lie between 0 and 1, and consider the expression

$$a^m+b^m+c^m+\ldots+k^m. \tag{1}$$

If a and b are unequal and we replace them by the equal numbers $\frac{1}{2}(a+b)$, $\frac{1}{2}(a+b)$, then the new expression obtained, viz.,

$$\{\tfrac{1}{2}(a+b)\}^m+\{\tfrac{1}{2}(a+b)\}^m+c^m+\ldots+k^m$$

is less than (1), since

$$2\{\tfrac{1}{2}(a+b)\}^m<a^m+b^m.$$

But the sum of the new numbers, viz., $\frac{1}{2}(a+b)+\frac{1}{2}(a+b)+c+\ldots+k$, is the same as before.

Thus keeping the sum of the terms constant, we can diminish the sum of their mth powers by replacing unequal terms by equal ones; and as long as (1) contains unequal terms, we can diminish it in the same way. Therefore, its least value is obtained when all the terms are equal. In this

case each term is $(a+b+c+\ldots+k)/n$, and the sum of the mth powers is

$$n\{(a+b+c+\ldots+k)/n\}^m.$$

Since this is the least value, it is less than (1), and so

$$\frac{\mathbf{a}^m+\mathbf{b}^m+\mathbf{c}^m+\ldots+\mathbf{k}^m}{\mathbf{n}}>\left(\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}+\ldots+\mathbf{k}}{\mathbf{n}}\right)^m.$$

If m lies between 0 and 1, we can show by a similar argument that the sign of inequality is reversed.

The above theorem can also be stated as follows: *The arithmetic mean of the mth powers of n positive numbers is greater than the mth power of their arithmetic mean, except when m lies between* 0 *and* 1.

If the numbers are all equal, or if $m=0$ or 1, the above inequality reduces to an equality.

Ex. Show that the sum of the mth powers of the first n even numbers is greater than $n(n+1)^m$ if $m>1$.

By the above theorem,

$$\frac{2^m+4^m+6^m+\ldots+(2n)^m}{n}>\left(\frac{2+4+6+\ldots+2n}{n}\right)^m,$$

or

$$2^m+4^m+6^m+\ldots+(2n)^m>n(n+1)^m,$$

on summing up the right-hand side.

3·6. An inequality for $(1+x/a)^a$. *If a, b and x are positive and $a>b$, then*

$$\left(1+\frac{x}{a}\right)^a>\left(1+\frac{x}{b}\right)^b.$$

(i) Let a and b be integers; then by expansion

$$\left(1+\frac{x}{a}\right)^a=1+a.\frac{x}{a}+\frac{a(a-1)}{2!}.\frac{x^2}{a^2}+\frac{a(a-1)(a-2)}{3!}.\frac{x^3}{a^3}+\cdots$$

$$=1+x+\left(1-\frac{1}{a}\right)\frac{x^2}{2!}+\left(1-\frac{1}{a}\right)\left(1-\frac{2}{a}\right)\frac{x^3}{3!}+\ldots, \quad (1)$$

and similarly

$$\left(1+\frac{x}{b}\right)^b=1+x+\left(1-\frac{1}{b}\right)\frac{x^2}{2!}+\left(1-\frac{1}{b}\right)\left(1-\frac{2}{b}\right)\frac{x^3}{3!}+\ldots. \quad (2)$$

Since $a>b$, the number of terms in the expansion (1) is greater than the number of terms in the expansion (2). Also each term in (1) is greater than the corresponding term in (2). Therefore

$$\left(1+\frac{x}{a}\right)^a>\left(1+\frac{x}{b}\right)^b.$$

(ii) If a and b are fractional*, we can write them as p/d and q/d, where p, q and d are positive integers. Then we have to prove that

$$\left(1+\frac{xd}{p}\right)^{p/d}>\left(1+\frac{xd}{q}\right)^{q/d},$$

or, raising both sides to the power d, we have to prove that

$$\left(1+\frac{xd}{p}\right)^p>\left(1+\frac{xd}{q}\right)^q.$$

This is true by (i), since $p>q$.

3·7. An inequality for $a^a b^b$. *To prove that*

$$(1+x)^{1+x}(1-x)^{1-x}>1 \text{ if } x<1,$$

and to deduce that $$a^a b^b>\left(\frac{a+b}{2}\right)^{a+b}.$$

*Only a rational number can be expressed as a ratio of two integers. Thus the theorem has been proved for rational exponents only. But if k is irrational, a^k is not an algebraic function, and "it is obvious that there can be no strictly algebraical proof." See Hardy, Littlewood and Polya: *Inequalities*.

Let $P=(1+x)^{1+x}(1-x)^{1-x}$; then

$$\log P=(1+x)\log(1+x)+(1-x)\log(1-x)$$
$$=\{\log(1+x)+\log(1-x)\}+x\{\log(1+x)-\log(1-x)\}$$
$$=-2\{\tfrac{1}{2}x^2+\tfrac{1}{4}x^4+\tfrac{1}{6}x^6+\ldots\}+2x\{x+\tfrac{1}{3}x^3+\tfrac{1}{5}x^5+\ldots\}$$
$$=2\left\{\frac{x^2}{1.2}+\frac{x^4}{3.4}+\frac{x^6}{5.6}+\ldots\right\},$$

which is positive. Therefore $P>1$, i.e.,

$$(1+x)^{1+x}(1-x)^{1-x}>1. \qquad (1)$$

To deduce the second result, put $x=(a-b)/(a+b)$, so that

$$1+x=2a/(a+b), \quad 1-x=2b/(a+b).$$

Then (1) becomes

$$\left(\frac{2a}{a+b}\right)^{2a/(a+b)}\left(\frac{2b}{a+b}\right)^{2b/(a+b)}>1,$$

or, raising both sides to the power $(a+b)/2$,

$$\left(\frac{2a}{a+b}\right)^{a}\left(\frac{2b}{a+b}\right)^{b}>1.$$

Multiplying both sides by $\{(a+b)/2\}^{a+b}$, we get

$$a^a b^b>\left(\frac{a+b}{2}\right)^{a+b}.$$

3·8. Miscellaneous Methods. We now give some examples illustrating methods other than those used above.

Ex. 1. Prove that if $1>x>y>0$,

$$\left(\frac{1+x}{1-x}\right)^{1/x}>\left(\frac{1+y}{1-y}\right)^{1/y}.$$

We have $\quad \left(\dfrac{1+x}{1-x}\right)^{1/x}>\text{ or }<\left(\dfrac{1+y}{1-y}\right)^{1/y}$

according as $\dfrac{1}{x}\log\left(\dfrac{1+x}{1-x}\right)>\text{ or }<\dfrac{1}{y}\log\left(\dfrac{1+y}{1-y}\right)$.

But $\dfrac{1}{x}\log\left(\dfrac{1+x}{1-x}\right)=\dfrac{1}{x}\{\log(1+x)-\log(1-x)\}$

$$=2\{1+\tfrac{1}{3}x^2+\tfrac{1}{5}x^4+\ldots\}, \qquad (1)$$

and similarly

$$\frac{1}{y}\log\left(\frac{1+y}{1-y}\right)=2\{1+\tfrac{1}{3}y^2+\tfrac{1}{5}y^4+\ldots\}. \qquad (2)$$

Since $x>y$, the expansion (1) is term by term greater than the expansion (2). Therefore

$$\frac{1}{x}\log\left(\frac{1+x}{1-x}\right)>\frac{1}{y}\log\left(\frac{1+y}{1-y}\right),$$

from which the proposition follows.

Ex. 2. If a, b, c are unequal positive integers, show that

$$\left(\frac{a^2+b^2+c^2}{a+b+c}\right)^{a+b+c}>a^a b^b c^c>\left(\frac{a+b+c}{3}\right)^{a+b+c}.$$

[*Banaras*, 1960]

(i) Take a factors each equal to a, b factors each equal to b, and c factors each equal to c. Then, since the arithmetic mean is greater than the geometric mean,

$$\frac{a.a+b.b+c.c}{a+b+c}>(a^a b^b c^c)^{1/(a+b+c)}.$$

Therefore $$\left(\frac{a^2+b^2+c^2}{a+b+c}\right)^{a+b+c}>a^a b^b c^c.$$

(ii) Take a factors equal to $1/a$, b factors equal to $1/b$, and c factors equal to $1/c$. Applying again the theorem that the arithmetic mean is greater than the geometric mean, we get

$$\frac{a(1/a)+b(1/b)+c(1/c)}{a+b+c}>\left(\frac{1}{a^a}\cdot\frac{1}{b^b}\cdot\frac{1}{c^c}\right)^{1/(a+b+c)},$$

or $$\left(\frac{3}{a+b+c}\right)^{a+b+c}>\frac{1}{a^a b^b c^c}.$$

Therefore $$a^a b^b c^c>\left(\frac{a+b+c}{3}\right)^{a+b+c}$$

EXAMPLES

1. Show that $8(a^2+b^2)(a^3+b^3)>(a+b)^5$.

2. If a, b, c are in harmonical progression and $n>1$, show that $a^n+c^n>2b^n$. [*Agra*, 1953]

3. Show that $16(a^3+b^3+c^3+d^3)>(a+b+c+d)^3$.

4. Prove that $n(n+1)^3<8(1^3+2^3+3^3+\ldots+n^3)$.

5. If $x_1^2+x_2^2+\ldots+x_n^2=a$, show that

$$na>(x_1+x_2+\ldots+x_n)^2>a. \qquad [\textit{Allahabad}, \text{'49}]$$

6. If a, b, c are unequal, prove that

$$\frac{2}{a+b}+\frac{2}{b+c}+\frac{2}{c+a}>\frac{9}{a+b+c}. \qquad [\textit{Rajasthan}, 1958]$$

7. Show that $\left(\dfrac{a+b+c+\ldots+k}{n}\right)^{a+b+c+\ldots+k}<a^ab^bc^c\ldots k^k$.

8. If $a_1, a_2, \ldots, a_n$ are positive numbers less than 1, show that $(1-a_1)(1-a_2)\ldots(1-a_n)>1-a_1-a_2-\ldots-a_n$.

EXAMPLES ON CHAPTER III

1. Under what circumstances is

$$\text{(i) } x^3+y^3>x^2y+xy^2; \text{ (ii) } (1+xy)^2<(x+y)^2?$$

2. If $x>0$, prove that $2\leqslant x+1/x\leqslant x^3+1/x^3$.

3. Show that $(1+x^3)(1+y^3)(1+z^3)>(1+xyz)^3$. [*Allahabad*, 1953]

4. If a, b, c are positive, show that

$$2(a^3+b^3+c^3)>bc(b+c)+ca(c+a)+ab(a+b).$$

5. Show that

$$(a+b+c+d)(a^3+b^3+c^3+d^3)>(a^2+b^2+c^2+d^2)^2.$$

[*U P. C. S.*, 1951]

6. If x, y, z are positive and unequal, show that

(i) $(x+y+z)^3>27(y+z-x)(z+x-y)(x+y-z)$.

(ii) $xyz>(y+z-x)(z+x-y)(x+y-z)$.

7. If $x+y+z=1$, show that the least value of $(1/x)+(1/y)+(1/z)$ is 9; and that $(1-x)(1-y)(1-z)>8xyz$. [*Rajasthan*, 1959]

8. If a, b, c are unequal and positive, prove that

$$\frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b}<\tfrac{1}{2}(a+b+c).$$ [*Agra*, 1955]

[Hint. Since $bc<\{\frac{1}{2}(b+c)\}^2$, therefore $bc/(b+c)<\frac{1}{4}(b+c)$. Write two more similar inequalities and add.]

9. Prove that

$$\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}+\frac{a^2+b^2}{a+b}\geqslant a+b+c.$$ [*Nagpur*, 1948]

10. If n be a positive integer greater than 2, show that

$$2^n>1+n\sqrt{2^{n-1}}.$$ [*U.P.C.S.*, 1960]

11. Show that

$$\left(\frac{x^2+y^2+z^2}{x+y+z}\right)^{x+y+z}>x^x y^y z^z,$$

unless $x=y=z$. [*Madras*, 1960]

12. Find the greatest value of $(7-x)^4(2+x)^5$ if x lies between 7 and -2. [*Kashmir*, 1953]

13. Find the maximum value of $x^{1/2}(1-x)^{1/3}$.

14. If α and β are positive and $\alpha>\beta$, show that

$$(1+1/\alpha)^\alpha>(1+1/\beta)^\beta.$$

Hence show that if $n>1$, the value of $(1+1/n)^n$ lies between 2 and 2·718.... [*Sagar*, 1949]

15. Show that $a^b b^a<\{\frac{1}{2}(a+b)\}^{a+b}$. [*Nagpur*, 1954]

16. If a, b, c are in descending order of magnitude, show that

$$\left(\frac{a+c}{a-c}\right)^a<\left(\frac{b+c}{b-c}\right)^b.$$ [*Rajputana* 1950]

17. If x is positive, show that

$$\log(1+x)<x \text{ and } >x/(1+x).$$

18. Show that

$\log_e(1+n)<1+\frac{1}{2}+\frac{1}{3}+\ldots+1/n.$ [*Allahabad*, 1949]

[Hint. First prove that $\log(1+r)-\log r<1/r$].

19. Prove that $(1/m)\log(1+a^m)<(1/n)\log(1+a^n)$ if $m>n$.

20. Show that $(x^m+y^m)^n<(x^n+y^n)^m$ if $m>n$.

[*Lucknow*, 1952]

21. If n is a positive integer and $x<1$, show that

$$\frac{1-x^{n+1}}{n+1}<\frac{1-x^n}{n}.$$ [*Rajasthan*, 1960]

22. If a, b, c denote the sides of a triangle, show that

(i) $a^2(p-q)(p-r)+b^2(q-r)(q-p)+c^2(r-p)(r-q)\geqslant 0$, p, q, r being any real numbers.

(ii) $a^2yz+b^2zx+c^2xy$ cannot be positive if $x+y+z=0$.

23. Show that $1!3!5!\ldots(2n-1)!>(n!)^n$. [*Mad.*, 1954]

24. If $a, b, c, d, \ldots$, are p positive integers, whose sum is equal to n, show that the least value of $a!\,b!\,c!\,d!\ldots$ is

$$\{q!\}^{p-r}\{(q+1)!\}^r,$$

where q is the quotient and r the remainder when n is divided by p. [*U.P.C.S.*, 1954]

CHAPTER IV

PARTIAL FRACTIONS

4·1. Partial Fractions. An expression of the form

$$\frac{a_0x^m+a_1x^{m-1}+\ldots+a_m}{b_0x^n+b_1x^{n-1}+\ldots+b_n},$$

in which $a_0, a_1, \ldots, b_0, b_1, \ldots$ are constants and m and n are positive integers, is called a *rational algebraic fraction.* If the numerator is of a lower degree than the denominator (i.e., if $m<n$), the fraction is called a *proper* fraction. If the numerator is not of a lower degree than the denominator, we can divide the numerator by the denominator and express the fraction as the sum of an integral part and a proper fraction. Thus

$$\frac{x^3+2x^2+1}{x^2-1}=x+2+\frac{x+3}{x^2-1}.$$

If the denominator of a proper fraction can be factorised, the fraction can be expressed as the sum of fractions with simpler denominators. Expressing a given fraction as a sum of simpler fractions is called resolving it into *partial fractions.* For example,

$$\frac{x+3}{x^2-1}=\frac{2}{x-1}-\frac{1}{x+1},$$

and so $2/(x-1)$ and $-1/(x+1)$ are the partial fractions of $(x+3)/(x^2-1)$.

To resolve a given proper fraction into partial fractions, the denominator should first be factor-

ised into its simplest possible real factors. These will be either linear or quadratic, and some of the factors may be repeated. The method of finding partial fractions corresponding to these factors is given in the next article. Unless mentioned otherwise, by resolution into partial fractions we shall mean the resolution into the simplest set of partial fractions.

4·2. Simplest set of Partial Fractions. *If the proper fraction $f(x)/\phi(x)$ be resolved as a sum of partial fractions, then,*

(i) *to every non-repeated linear factor $x-a$ in $\phi(x)$ corresponds a partial fraction of the form $A/(x-a)$;*

(ii) *to every linear factor repeated r times in $\phi(x)$, i.e., to every factor $(x-b)^r$, corresponds a group of r partial fractions of the form*

$$\frac{B_1}{x-b}+\frac{B_2}{(x-b)^2}+\frac{B_3}{(x-b)^3}+\cdots+\frac{B_r}{(x-b)^r};$$

(iii) *to every non-repeated quadratic factor x^2+px+q in $\phi(x)$ corresponds a partial fraction of the form*

$$(Cx+D)/(x^2+px+q);$$

(iv) *to every quadratic factor repeated s times in $\phi(x)$, i.e., to every factor $(x^2+kx+l)^s$, corresponds a group of s partial fractions of the form*

$$\frac{E_1x+F_1}{x^2+kx+l}+\frac{E_2x+F_2}{(x^2+kx+l)^2}+\cdots+\frac{E_sx+F_s}{(x^2+kx+l)^s}.$$

Here the coefficients $A, B_1, B_2, \ldots, C, D, E_1, F_1, \ldots$ are independent of x.

To prove the above, we notice that the number of constants $A, B_1, B_2, \ldots$ is n, where n is the degree of the denominator $\phi(x)$; the reason being that for every linear

factor we have one constant, for every quadratic factor we have two constants, for a linear factor repeated r times, viz. $(x-b)^r$, we have r constants, and so on.

Also, if we equate $f(x)/\phi(x)$ to the sum of partial fractions indicated by the above rules, and multiply both sides by $\phi(x)$ to get rid of the denominators, we obtain $f(x)$ on the left-hand side and an expression of degree $n-1$ on the right-hand side. Since this relation has to be satisfied for all values of x, it must be an identity. Equating, therefore, the coefficients of the various powers of x and the constant terms on the two sides, weo btain n equations—exactly as many as there are constants at our disposal. On solving, these equations will give values for the n constants A, B_1, B_2,... uniquely*, thus proving our hypothesis.

4·3. Resolution into Partial Fractions. We see that all the partial fractions can be obtained by applying the following rules :

(i) If the given fraction, say $F(x)/\phi(x)$, is not a proper fraction, divide $F(x)$ by $\phi(x)$ till the remainder, say $f(x)$, is of a lower degree than $\phi(x)$. If the quotient is $Q(x)$, then

$$\frac{F(x)}{\phi(x)}=Q(x)+\frac{f(x)}{\phi(x)}.$$

(ii) Resolve $\phi(x)$ into its real prime factors.

(iii) Equate $f(x)/\phi(x)$ to a sum of suitable partial fractions in accordance with the rules of the preceding article, and multiply both the sides by $\phi(x)$.

(iv) To determine the constants occurring in this identity, equate, respectively, the coefficients of the various powers of x on one side to the coefficients of the same powers of x on the other side, and solve the set of simultaneous equations so obtained.

* *Uniquely* means that there will be only one value for each constant. This point really requires proof, but that is beyond our scope.

In actual practice the simpler rules given in the subsequent articles are preferred, because they save labour in the numerical computations.

Ex. Resolve into partial fractions

$$\frac{x^3-5}{(x-1)(x+1)(x-2)}.$$

Since the numerator is of the same degree as the denominator, we get on division

$$\frac{x^3-5}{(x-1)(x+1)(x-2)}=1+\frac{2x^2+x-7}{(x-1)(x+1)(x-2)}.$$

Put $\dfrac{2x^2+x-7}{(x-1)(x+1)(x-2)}=\dfrac{A}{x-1}+\dfrac{B}{x+1}+\dfrac{C}{x-2}$,

and multiply both sides by $(x-1)(x+1)(x-2)$. Then we get

$$2x^2+x-7=A(x+1)(x-2)+B(x-1)(x-2)+C(x-1)(x+1).$$

Equating the coefficients of x^2, x and the constant terms on the two sides, we obtain

$$2=A+B+C,\ 1=-A-3B,\ -7=-2A+2B-C.$$

On solving, these give $A=2$, $B=-1$, $C=1$. Therefore

$$\frac{x^3-5}{(x-1)(x+1)(x-2)}=1+\frac{2}{x-1}-\frac{1}{x+1}+\frac{1}{x-2}.$$

4·31. A Simple Rule. The partial fractions corresponding to non-repeated linear factors in the denominator of the fraction to be broken up can be easily found by the simple rule derived below.

Suppose we have to resolve $f(x)/\phi(x)$ into partial fractions, and there is a non-repeated linear factor $x-a$ in the denominator $\phi(x)$. Let $\phi(x)=(x-a)\psi(x)$. Then the given fraction, i.e.,

$\dfrac{f(x)}{(x-a)\psi(x)}=\dfrac{A}{x-a}+$ partial fractions not containing $x-a$ in the denominator.

Multiplying both sides by $x-a$, we have

$$f(x)/\psi(x)=A+(x-a)\times\text{partial fractions not containing } x-a \text{ in the denominator.}$$

Putting $x=a$ in this identity, we get

$$f(a)/\psi(a)=A,$$

so that
$$\frac{A}{x-a}=\frac{f(a)}{(x-a)\psi(a)}.$$

Since the right-hand side can be obtained from $f(x)/(x-a)\psi(x)$ by putting in it $x=a$ everywhere except in the factor $x-a$, we get the rule:

To obtain the partial fraction corresponding to the factor $x-a$ in the denominator, put $x=a$ everywhere in the given fraction except in the factor $x-a$ itself.

We can, similarly, deduce the rule : *to obtain the partial fraction $B_r/(x-b)^r$, corresponding to the factor $(x-b)^r$ in the denominator, put $x=b$ everywhere in the given fraction except in the factor $(x-b)^r$.* It should be noted that this rule only gives $B_r/(x-b)^r$. The other partial fractions corresponding to $(x-b)^r$ should be found by other methods.

A little consideration will show that the rules are true even if the given fraction is not a proper fraction.

Ex. 1. Solve by this method the example of the preceding article.

Comparing the coefficients of x^3 in the numerator and the denominator, we see that the integral part will be 1. As regards the partial fractions, we can apply the rule for non-repeated linear factors. Hence the given fraction is equal to

$$=1+\frac{2\,.\,1^2+1-7}{(x-1)(1+1)(1-2)}+\frac{2(-1)^2-1-7}{(-1-1)(x+1)(-1-2)}$$
$$+\frac{2\,.\,2^2+2-7}{(2-1)(2+1)(x-2)}$$
$$=1+\frac{2}{x-1}-\frac{1}{x+1}+\frac{1}{x-2}.$$

4·32. By substitution of Particular Values. When the denominator of the fraction to be broken up contains non-repeated linear factors only, all the partial fractions can be obtained by the rule of the preceding article. But when the denominator contains other factors also, additional equations are necessary to determine the corresponding partial fractions.

Although these equations can be obtained as in § 4·3, by equating to zero the coefficients of the various powers of x in the identity obtained in step (iii) of § 4·3, the alternative method of obtaining equations by substituting successively convenient values of x in the above identity is easier. This method is particularly suitable when only one or two constants remain to be determined.

We may, moreover, apply this method to avoid the labour of division when the numerator is not of a lower degree than the denominator, as in the example below.

Ex. Resolve into partial fractions $9x^4/(x-1)(x+2)^2$.

We see mentally that the first term of the quotient will be $9x$; let us assume, therefore, that

$$\frac{9x^4}{(x-1)(x+2)^2}=9x+A+\frac{B}{x-1}+\frac{C}{x+2}+\frac{D}{(x+2)^2}. \quad (1)$$

Then, by our rule (§ 4·3),

$B=9\,.\,1^4/3^2=1$, and $D=9(-2)^4/(-2-1)=-48$.

Now putting $x=0$ in (1), we see that $0=A-B+\frac{1}{2}C+\frac{1}{4}D$,

i.e., $$A+\tfrac{1}{2}C=13.$$

Similarly, putting $x=-1$, $A+C=53$.

Solving these two, $A=-27$, and $C=80$.

Thus the given fraction

$$=9x-27+\frac{1}{x-1}+\frac{80}{x+2}-\frac{48}{(x+2)^2}.$$

NOTE: In the case of proper fractions it is often convenient to obtain one of the equations for determining the unknown constants by letting $x\to\infty$ (after multiplying out by x) instead of giving a finite value to x.

EXAMPLES

Resolve into partial fractions :

1. $(5x+5)/(x-1)(x+4)$.
2. $(7x-1)/(1-5x+6x^2)$.
3. $(x^2+1)/(x^2-1)$.
4. $(1+3x+2x^3)/(1-2x)(1-x^2)$.
5. $\dfrac{x^2-10x+13}{(x-1)(x^2-5x+6)}$.
6. $\dfrac{2x^2-11x+5}{(x-3)(x^2+2x-5)}$. [*Agra*, 1950]
7. $\dfrac{x^3}{(x-a)(x-b)(x-c)}$.
8. $\dfrac{x^2-x+1}{(x+1)(x-1)^2}$. [*Utkal*, '49]
9. $\dfrac{x^5+3}{x(x^2-1)(x+1)}$.
10. $\dfrac{4+7x}{(2+3x)(1+x)^2}$. [*Lkw.*, '55]

4·33. Long Division for Repeated Factors. When the denominator of a given fraction contains repeated linear factors, we can apply the following method to find the corresponding partial fractions.

Let the given fraction be $f(x)/\phi(x)$ and let $\phi(x) = (x-a)^r\psi(x)$. Put $x-a=y$; then

$$\frac{f(x)}{\phi(x)} = \frac{f(x)}{(x-a)^r\psi(x)} = \frac{f(a+y)}{y^r\psi(a+y)}$$

$$= \frac{1}{y^r}\cdot\frac{A_0+A_1y+A_2y^2+\dots}{B_0+B_1y+B_2y^2+\dots},$$

say, when the numerator and denominator are arranged in ascending powers of y. Now divide $A_0+A_1y+\dots$ by $B_0+B_1y+\dots$ and continue the process till y^r becomes a common factor of the remainder. Suppose the quotient is

$$C_0+C_1y+\dots+C_{r-1}y^{r-1},$$

and the remainder is $v^r(D_0+D_1y+\dots)$, so that

$$\frac{A_0+A_1y+A_2y^2+\dots}{B_0+B_1y+B_2y^2+\dots}=C_0+C_1y+\dots+C_{r-1}y^{r-1}$$
$$+\frac{y^r(D_0+D_1y+\dots)}{B_0+B_1y+\dots}.$$

Then
$$\frac{f(x)}{\phi(x)}=\frac{C_0}{y^r}+\frac{C_1}{y^{r-1}}+\dots+\frac{C_{r-1}}{y}+\frac{D_0+D_1y+\dots}{B_0+B_1y+\dots}$$
$$=\frac{C_0}{(x-a)^r}+\frac{C_1}{(x-a)^{r-1}}+\dots+\frac{C_{r-1}}{x-a}$$
$$+\frac{D_0+D_1(x-a)+\dots}{\psi(x)}.$$

Thus the partial fractions corresponding to $(x-a)^r$ have been determined. The fraction
$$\{D_0+D_1(x-a)+\dots\}/\psi(x)$$
can now be further resolved into partial fractions by the methods of the present article or of the preceding articles, as the case may be.

NOTE. The present method is convenient when a linear factor is repeated several times in the denominator.

Ex. Resolve into partial fractions
$$\frac{6+13x-3x^3}{(x-1)(x+1)^3(x+2)}.$$

Put $x+1=y$; then
$$\frac{6+13x-3x^3}{(x+1)^3(x-1)(x+2)}=\frac{6+13(y-1)-3(y-1)^3}{y^3(y-2)(y+1)}$$
$$=\frac{1}{y^3}\cdot\frac{-4+4y+9y^2-3y^3}{-2-y+y^2}.$$

Now divide the numerator by $-2-y+y^2$, till y^3 is a factor of the remainder, as shown below:

$$\begin{array}{l}
-2-y+y^2)\,-4+4y+9y^2-3y^3\,(2-3y-2y^2\\
\qquad\qquad\quad \underline{-4-2y+2y^2\qquad\qquad}\\
\qquad\qquad\qquad\quad 6y+7y^2-3y^3\\
\qquad\qquad\qquad\quad \underline{6y+3y^2-3y^3}\\
\qquad\qquad\qquad\qquad\quad 4y^2\\
\qquad\qquad\qquad\qquad\quad \underline{4y^2+2y^3-2y^4}\\
\qquad\qquad\qquad\qquad\quad\;\; -2y^3+2y^4
\end{array}$$

Then the given fraction

$$=\frac{1}{y^3}\left\{2-3y-2y^2+\frac{y^3(-2+2y)}{-2-y+y^2}\right\}$$

$$=\frac{2}{y^3}-\frac{3}{y^2}-\frac{2}{y}+\frac{2y-2}{(y-2)(y+1)}$$

$$=\frac{2}{(x+1)^3}-\frac{3}{(x+1)^2}-\frac{2}{x+1}+\frac{2x}{(x-1)(x+2)}.$$

Also, by the rule for non-repeated linear factors,

$$\frac{2x}{(x-1)(x+2)}=\frac{2.1}{(x-1)(1+2)}+\frac{2(-2)}{(-2-1)(x+2)}.$$

Therefore $\dfrac{6+13x-3x^3}{(x-1)(x+2)(x+1)^3}$

$$=\frac{2}{3(x-1)}+\frac{4}{3(x+2)}-\frac{2}{x+1}-\frac{3}{(x+1)^2}+\frac{2}{(x+1)^3}.$$

4·34. Quadratic Factors. The partial fractions corresponding to quadratic factors in the denominator can be found by the method of § 4·3. But when the denominator contains two or more non-repeated quadratic factors, the following method will prove simpler.

Let the given fraction be $f(x)/\phi(x)$, and let

$$\phi(x)=(x^2+px+q)\psi(x).$$

Then we may put

$$\frac{f(x)}{\phi(x)}=\frac{Ax+B}{x^2+px+q}+\text{partial fractions not containing } x^2+px+q \text{ in the denominator.}$$

Multiplying by $\phi(x)$, we get

$$f(x)=(Ax+B)\psi(x)+(x^2+px+q)\psi(x)\times \text{ partial fractions not containing } x^2+px+q \text{ in the denominator.} \quad (1)$$

If we put $\qquad x^2=-px-q \qquad . \ . \ . \quad (2)$

in this, the second term on the right of (1) vanishes. Also with the help of (2), $f(x)$ and $(Ax+B)\psi(x)$ can successively

be reduced to linear expressions. This new linear relation is also an identity, as it has to be satisfied by the *two* roots of (2). Therefore, equating the coefficients of x and the constant terms on the two sides, we get two equations from which A and B can be determined.

Ex. 1. Resolve into partial fractions

$$(2x+1)/(x-1)(x^2+1).$$

Let $$\frac{2x+1}{(x-1)(x^2+1)}=\frac{A}{x-1}+\frac{Bx+C}{x^2+1},$$

and multiply both the sides by $(x-1)(x^2+1)$; then

$$2x+1=A(x^2+1)+(Bx+C)(x-1).$$

Putting $x=1$ in this, we get $A=\frac{3}{2}$; and equating the coefficients of x^2 and the constant terms on the two sides, we get

$$0=A+B \quad \text{and} \quad 1=A-C,$$

which give $$B=-\tfrac{3}{2}, C=\tfrac{1}{2}.$$

Therefore $$\frac{2x+1}{(x-1)(x^2+1)}=\frac{1}{2}\left\{\frac{3}{x-1}-\frac{3x-1}{x^2+1}\right\}.$$

Ex. 2. Resolve into partial fractions

$$\frac{2x^3+2x^2+4x+1}{(x^2+1)(x^2+x+1)}.$$ [*Alld.*, 1959]

Let $$\frac{2x^3+2x^2+4x+1}{(x^2+1)(x^2+x+1)}=\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+x+1};$$

then

$$2x^3+2x^2+4x+1=(Ax+B)(x^2+x+1)+(Cx+D)(x^2+1). \quad (1)$$

Putting $x^2=-x-1$ in this, we get

$$2x(-x-1)+2(-x-1)+4x+1=(Cx+D)(-x-1+1),$$

or $$-2x^2-1=-Cx^2-Dx,$$

or $$-2(-x-1)-1=-C(-x-1)-Dx,$$

or $$2x+1=(C-D)x+C.$$

This is an identity in x; therefore

$$2=C-D \text{ and } 1=C,$$

which give $C=1$ and $D=-1$.

Again, putting $x^2=-1$ in (1) we get

$$-2x-2+4x+1=(Ax+B)(-1+x+1),$$

or $$2x-1=Ax^2+Bx,$$

or $$2x-1=-A+Bx.$$

Therefore $A=1$ and $B=2$.

Hence the given fraction

$$=\frac{x+2}{x^2+1}+\frac{x-1}{x^2+x+1}.$$

EXAMPLES

Resolve into partial fractions :

1. $\dfrac{3x^3-8x^2+10}{(x-1)^4}$.
2. $\dfrac{2x^3}{(x-1)^3(x+4)}$. [*Annam.*, '49]
3. $\dfrac{5x^3+6x^2+5x}{(x^2-1)(x+1)^3}$. [*Nag.*, '54]
4. $\dfrac{x^4}{(x-1)^4(x+1)}$. [*Agra*, 1951]
5. $\dfrac{x+1}{(x-1)^2(x+2)^2}$.
6. $\dfrac{x^2+x}{(x-1)^2(x^2+4)}$. [*Alld.*, '49]
7. $\dfrac{2x^2+3x+4}{(x+1)(x^2+4)}$.
8. $\dfrac{3x^2+3}{(x^2-1)(x^2+x+1)}$.
9. $\dfrac{x^3+x^2+1}{(x^2+2)(x^2+3)}$.
10. $\dfrac{x^3+5x^2+x}{(x+1)(x^2+1)(x^3+1)}$.

4·4. Expansion of Rational Fractions. A rational fraction can be expanded in ascending powers of x by first resolving the fraction into partial fractions, each of which can then be expanded by the Binomial Theorem.

Ex. Find the general term in the expansion of

$$\frac{2x+1}{(x-1)(x^2+1)}$$

in ascending powers of x. [*Lucknow*, 1956]

By Ex. 1 of the preceding article

$$\frac{2x+1}{(x-1)(x^2+1)}=\frac{1}{2}\left\{\frac{3}{x-1}-\frac{3x-1}{x^2+1}\right\}$$

$$=-\tfrac{3}{2}(1-x)^{-1}-\tfrac{1}{2}(3x-1)(1+x^2)^{-1}$$

$$=-\tfrac{3}{2}(1+x+x^2+\ldots+x^p+\ldots)$$

$$+(\tfrac{1}{2}-\tfrac{3}{2}x)\{1-x^2+x^4-\ldots+(-1)^p x^{2p}+\ldots\}.$$

Therefore, the term in x^{2p} is

$$-\tfrac{3}{2}x^{2p}+\tfrac{1}{2}(-1)^p x^{2p}=\tfrac{1}{2}\{-3+(-1)^p\}x^{2p},$$

and the term in x^{2p+1} is

$$-\tfrac{3}{2}x^{2p+1}-\tfrac{3}{2}x(-1)^p x^{2p}=\tfrac{3}{2}\{-1+(-1)^{p+1}\}x^{2p+1}.$$

Hence the general term is

$$\tfrac{1}{2}\{-3+(-1)^{r/2}\}x^r \text{ when } r \text{ is even,}$$

and

$$\tfrac{3}{2}\{-1+(-1)^{(r+1)/2}\}x^r \text{ when } r \text{ is odd.}$$

4·5. Summation of Series. When the terms of a given series are rational fractions, which can be expressed as a difference of two (or more) fractions in such a way that on addition the fractions in successive terms cancel, the series can be easily summed up.

It should be noticed that the terms are not to be resolved into the simplest set of partial fractions, but are to be expressed as a difference of two (or more) suitable fractions.

Ex. Find the sum of n terms of the series

$$\frac{1}{(1+x)(1+x^2)}+\frac{x}{(1+x^2)(1+x^3)}+\frac{x^2}{(1+x^3)(1+x^4)}+\ldots.$$

The rth term, say u_r, is

$$\frac{x^{r-1}}{(1+x^r)(1+x^{r+1})}=\frac{z}{(1+xz)(1+x^2z)},$$

where we have put $x^{r-1}=z$.

Treating this as a fraction in z, we get by the rule for non-repeated linear factors

$$\frac{z}{(1+xz)(1+x^2z)}=\frac{(-1/x)}{(1+xz)(1-x)}+\frac{(-1/x^2)}{(1-1/x)(1+x^2z)}$$
$$=\frac{1}{x(x-1)}\left(\frac{1}{1+xz}-\frac{1}{1+x^2z}\right),$$

i.e.,
$$u_r=\frac{1}{x(x-1)}\left(\frac{1}{1+x^r}-\frac{1}{1+x^{r+1}}\right).$$

Therefore $u_1+u_2+\dots+u_n$

$$=\frac{1}{x(x-1)}\left\{\left(\frac{1}{1+x}-\frac{1}{1+x^2}\right)+\left(\frac{1}{1+x^2}-\frac{1}{1+x^3}\right)+\dots\right.$$
$$\left.+\left(\frac{1}{1+x^n}-\frac{1}{1+x^{n+1}}\right)\right\}$$

$$=\frac{1}{x(x-1)}\left\{\frac{1}{1+x}-\frac{1}{1+x^{n+1}}\right\}.$$

EXAMPLES ON CHAPTER IV

Resolve into partial fractions :

1. $(x^2+1)/x(x^2-1)$.
2. $x/(x-a)(x-b)(x-c)$.
3. $\dfrac{3x^2+x-2}{(x-2)^2(1-2x)}$. [*Nagpur*, 1950]
4. $\dfrac{x^4-3x^3-3x^2+10}{(x+1)^2(x-3)}$. [*Allahabad*, 1960]
5. $(5-9x)/(1-3x)^3(1+x)$. [*Madras*, 1960]
6. $x^3/(x-1)^4(x^2-x+1)$. [*Sagar*, 1951]
7. $\dfrac{x^2-x+1}{(x-1)^2(x-2)(x^2+1)}$. [*Rajasthan*, 1957]
8. $x^3/(x+2)^3(x^2+2)$. [*Utkal*, 1952]
9. $1/(x^4+1)$. [*Gorakhpur*, 1960]
10. $(x^2+1)^2/(x^4+x^2+1)$. [*Lucknow Prel.*, 1955]

Find the coefficient of x^r when the following expressions are expanded in ascending powers of x :

11. $\dfrac{5}{3-x-2x^2}$. [*Patna*, 1949]

12. $\dfrac{2x-4}{(1-x^2)(1-2x)}$.

13. $\dfrac{7+x}{(1+x)(1+x^2)}$. [*Allahabad*, 1946]

14. $\dfrac{4+7x}{(2+3x)(1+x)^2}$.

15. $\dfrac{3+2x-x^2}{(1+x)(1-4x)^2}$. [*Sagar*, 1948]

16. $\dfrac{x^2+2}{(x+1)^2(x+2)(x+3)}$. [*Annamalai*, 1950]

17. $\dfrac{1}{(1-ax)(1-bx)(1-cx)}$. [*Allahabad*, 1957]

18. Break $(x-2)/(x+2)(x-1)^2$ into partial fractions and show that the coefficient of x^n in the Binomial expansion is

$$\tfrac{4}{9}(-\tfrac{1}{2})^{n+1}-\tfrac{1}{9}(3n+7).$$ [*Rajputana*, 1949]

19. When $0<x<1$, find the sum to infinity of the series

$$\frac{1}{(1-x)(1-x^3)}+\frac{x^2}{(1-x^3)(1-x^5)}+\frac{x^4}{(1-x^5)(1-x^7)}+\dots.$$

[*Patna*, 1958]

20. Find the sum of n terms of the series

$$\frac{x(1-ax)}{(1+x)(1+ax)(1+a^2x)}+\frac{ax(1-a^2x)}{(1+ax)(1+a^2x)(1+a^3x)}+\dots.$$

CHAPTER V

RECURRING SERIES

5·1. Recurring Series. Let

$$a_0+a_1x+a_2x^2+\ldots+a_nx^n+\ldots \qquad (1)$$

be a series in which any $m+1$ successive coefficients are connected by the equation

$$a_n+p_1a_{n-1}+p_2a_{n-2}+\ldots+p_ma_{n-m}=0, \qquad (2)$$

where m is a fixed number and $p_1, p_2, \ldots, p_m$ are constants. Such a series is called a *recurring series.* The equation (2) connecting the successive coefficients is called the *scale of relation* of the recurring series.

In particular, the series (1) is a recurring series if the successive coefficients are connected by the scale of relation

$$a_n+p_1a_{n-1}+p_2a_{n-2}=0.$$

Thus the series

$$1+3x+5x^2+7x^3+9x^4+\ldots$$

is a recurring series, for

$$5-2\,.\,3+1=0,\ 7-2\,.\,5+3=0,\ 9-2\,.\,7+5=0, \ldots,$$

and in general, the coefficients satisfy the relation

$$a_n-2a_{n-1}+a_{n-2}=0.$$

When the scale of relation and a sufficient number of terms in the beginning of a recurring series are given, we can find as many terms of the series as we like. Thus, if the scale of relation is

$$a_n-3a_{n-1}+2a_{n-2}=0, \qquad (3)$$

and the first two terms are 1 and $3x$, then, by (3),

$$a_2=3a_1-2a_0=3\,.\,3-2\,.\,1=7,$$
$$a_3=3a_2-2a_1=3\,.\,7-2\,.\,3=15,$$
$$a_4=3a_3-2a_2=3\,.\,15-2\,.\,7=31,\ldots;$$

and the recurring series is

$$1+3x+7x^2+15x^3+31x^4+\ldots.$$

Instead of calling (2) the scale of relation of the recurring series (1), some authors use the term 'scale of relation' to denote the polynomial

$$1+p_1x+p_2x^2+p_3x^3+\ldots+p_mx^m.$$

But this definition is inconvenient when dealing with the recurring series

$$a_0+a_1+a_2+a_3+\ldots,$$

obtained by putting $x=1$ in (1). Unless mentioned specifically, we shall use the term in its former sense.

Ex. Show that the series $\Sigma(5+3^n)x^n$ is a recurring series.

Here $\quad a_n=5+3^n$ and $a_{n-1}=5+3^{n-1}$.

Therefore $$a_n-a_{n-1}=2\,.\,3^{n-1}. \tag{1}$$

Putting $n-1$ for n in this, we have

$$a_{n-1}-a_{n-2}=2\,.\,3^{n-2}. \tag{2}$$

Multiplying (2) by 3 and subtracting from (1), we get

$$a_n-a_{n-1}-3(a_{n-1}-a_{n-2})=0.$$

i.e., $$a_n-4a_{n-1}+3a_{n-2}=0,$$

showing that the given series is a recurring series.

5·2. Scale of Relation. When a sufficient number of terms of a recurring series are given, the scale of relation can be determined from them.

To find how many terms are required to determine the scale of relation, suppose that it contains two constants. Then we can assume it to be

$$a_n+p_1a_{n-1}+p_2a_{n-2}=0.$$

To determine p_1 and p_2, we can form the equations
$$a_2+p_1a_1+p_2a_0=0 \quad \text{and} \quad a_3+p_1a_2+p_2a_1=0.$$
Therefore, four consecutive terms are required to determine a scale of relation containing two constants.

Similarly, if the scale of relation contains m constants, we can assume it to be
$$a_n+p_1a_{n-1}+p_2a_{n-2}+\dots+p_ma_{n-m}=0.$$
To determine $p_1, p_2, \dots, p_m$, m equations are required. The first of these equations will involve $m+1$ terms of the given series. The remaining $m-1$ equations will each involve one additional term. Thus a total of $(m+1)+(m-1)$, i.e. $2m$, consecutive terms of the recurring series will be required to determine a scale of relation containing m constants.

Conversely, if $2m$ consecutive terms of a recurring series are given, we may assume the scale of relation to be
$$a_n+p_1a_{n-1}+p_2a_{n-2}+\dots+p_ma_{n-m}=0,$$
and determine p_1, p_2, ... from the m equations which can be formed. If the scale of relation contains fewer constants than m, then one or more of the constants p_m, $p_{m-1}, \dots$ will come out to be zero.

If $2m+1$ consecutive terms of a series are given, and the series is known to be a recurring one, we can assume the scale of relation as above. But now $m+1$ equations connecting p_1, p_2, ... can be written down. Any m of these can be used to determine p_1, p_2, ...; and the remaining equation will be satisfied by the values determined from the other equations.

NOTE. If the first $2m$ terms of a series are assigned any arbitrary values, we can continue the series as a recurring series by finding a scale of relation by the above method.

But we cannot take $2m+1$ terms at random: the $(2m+1)$th term must satisfy the scale of relation derived from the first $2m$ terms.

Ex. Find the scale of relation of the series

$$2+3x+5x^2+9x^3+\dots.$$

Let the scale of relation be

$$a_n+pa_{n-1}+qa_{n-2}=0;$$

then, since the given terms must satisfy it, we have

$$5+3p+2q=0 \text{ and } 9+5p+3q=0.$$

Solving these we get $p=-3$ and $q=2$.

Therefore the scale of relation is

$$a_n-3a_{n-1}+2a_{n-2}=0.$$

5·3. Sum of n terms. *To find the sum of n terms of a recurring series whose scale of relation is*

$$a_n+pa_{n-1}+qa_{n-2}=0. \tag{1}$$

Let the recurring series be $a_0+a_1x+a_2x^2+\dots$, and let S_n denote the sum of the first n terms; then

$$S_n=a_0+a_1x+a_2x^2+\dots+a_{n-1}x^{n-1},$$
$$pxS_n=pa_0x+pa_1x^2+\dots+pa_{n-2}x^{n-1}+pa_{n-1}x^n,$$
$$qx^2S_n=qa_0x^2+\dots+qa_{n-3}x^{n-1}+qa_{n-2}x^n+qa_{n-1}x^{n+1}.$$

Adding these, we get

$$(1+px+qx^2)S_n=a_0+(a_1+pa_0)x$$
$$+(pa_{n-1}+qa_{n-2})x^n+qa_{n-1}x^{n+1},$$

the coefficient of every other power of x being zero by (1). Therefore

$$S_n=\frac{a_0+(a_1+pa_0)x}{1+px+qx^2}+\frac{(pa_{n-1}+qa_{n-2})x^n+qa_{n-1}x^{n+1}}{1+px+qx^2}.$$

A similar method is applicable to series whose scale of relation contains more than two constants.

5·4. Generating Function. Let $a_0+a_1x+a_2x^2+\ldots$ be a recurring series whose scale of relation is

$$a_n+pa_{n-1}+qa_{n-2}=0. \tag{1}$$

Suppose only p, q, a_0 and a_1 are given. Then we see by actual division that

$$\frac{a_0+(a_1+pa_0)x}{1+px+qx^2}=a_0+a_1x-\frac{(pa_1+qa_0)x^2+qa_1x^3}{1+px+qx^2}$$

$$=a_0+a_1x+\frac{a_2x^2-qa_1x^3}{1+px+qx^2}, \text{ by (1)},$$

$$=a_0+a_1x+a_2x^2-\frac{(pa_2+qa_1)x^3+qa_2x^4}{1+px+qx^2}=\ldots$$

$$=a_0+a_1x+a_2x^2+\ldots+a_{n-1}x^{n-1}$$
$$-\frac{(pa_{n-1}+qa_{n-2})x^n+qa_{n-1}x^{n+1}}{1+px+qx^2}. \tag{2}$$

Thus we can obtain any number of terms of the recurring series by simply dividing out $a_0+(a_1+pa_0)x$ by $1+px+qx^2$.

This process really amounts to expanding

$$\frac{a_0+(a_1+pa_0)x}{1+px+qx^2} \tag{3}$$

in ascending powers of x, and the expansion can be carried out by any other suitable method, as in the examples below. Since the expression (3) enables us to obtain the whole recurring series, it is known as the *generating function* of the recurring series.

It is seen from (2) that the generating fuuction is equivalent to the infinite series $a_0+a_1x+a_2x^2+\ldots$ only when the remainder after n terms, viz.,

$$\frac{(pa_{n-1}+qa_{n-2})x^n+qa_{n-1}x^{n+1}}{1+px+qx^2},$$

tends to zero as n tends to infinity; that is, only when the recurring series is convergent. In this case the generating function and the sum of the recurring series are identical. It can be shown that every recurring series is convergent for sufficiently small values of x.

When the recurring series is not convergent, its sum has no meaning, and the generating function is just a formal expression which can be used to obtain the terms of the recurring series.

5·5. Application to Examples. To obtain the generating function of a given series we may use the formula (3) above, or we may suppose the series to be convergent and obtain the sum to infinity of the recurring series by the method of § 5·3.

Sometimes we are given the first few terms of a recurring series and are required to find the sum of the first n terms. For this purpose, first the scale of relation should be found and then the generating function. The latter gives on expansion the general terms, after which the sum of n terms can be found by the method of § 5·3.

When dealing with the series $a_0+a_1+a_2+\ldots$, which does not involve x, we must find first the generating function of the series $a_0+a_1x+a_2x^2+\ldots$. After expansion x can be put equal to unity.

Ex. 1. Find the sum to n terms of the series

$$2+3x+5x^2+9x^3+\ldots .$$

By Ex. § 5·2, the scale of relation of this series is

$$a_n-3a_{n-1}+2a_{n-2}=0.$$

Therefore, if the generating function is S, we have

$$\begin{aligned} S&=2+3x+5x^2+9x^3+\ldots,\\ -3xS&=\quad\ -6x-9x^2-15x^3-\ldots,\\ 2x^2S&=\qquad\qquad\ 4x^2+\ 6x^3+\ldots . \end{aligned}$$

Adding and dividing by $1-3x+2x^2$, we get

$$S=\frac{2-3x}{1-3x+2x^2}.$$

On resolving into partial fractions, we see that

$$S=\frac{2-3x}{(1-x)(1-2x)}=\frac{1}{1-x}+\frac{1}{1-2x},$$

and the general term is easily seen on expansion by the Binomial Theorem to be $(1+2^n)x^n$.

The sum to n terms can now be found by § 5·3, or more simply as follows. The first n terms

$$=(1+x+x^2+\ldots+x^{n-1})+(1+2x+2^2x^2+\ldots+2^{n-1}x^{n-1})$$

$$=\frac{1-x^n}{1-x}+\frac{1-2^nx^n}{1-2x}.$$

Ex. 2. Sum the following recurring series up to n terms:

$$1+13+7+10+\ldots.$$

Let the scale of relation be $a_n+pa_{n-1}+qa_{n-2}=0$; then

$$7+13p+q=0 \text{ and } 10+7p+13q=0,$$

giving $$p=-\tfrac{1}{2} \text{ and } q=-\tfrac{1}{2}.$$

Consider now the series $1+13x+7x^2+10x^3+\ldots$, and let its generating function be S; then

$$S=1+13x+7x^2+10x^3+\ldots,$$

$$-\tfrac{1}{2}xS=\quad -\tfrac{1}{2}x-\tfrac{13}{2}x^2-\tfrac{7}{2}x^3-\ldots,$$

$$-\tfrac{1}{2}x^2S=\quad\quad -\tfrac{1}{2}x^2-\tfrac{13}{2}x^3-\ldots.$$

Therefore $(1-\frac{1}{2}x-\frac{1}{2}x^2)S=1+\frac{25}{2}x$,

or $$S=\frac{1+\frac{25}{2}x}{1-\frac{1}{2}x-\frac{1}{2}x^2}=\frac{1+\frac{25}{2}x}{(1-x)(1+\frac{1}{2}x)}=\frac{9}{1-x}-\frac{8}{1+\frac{1}{2}x},$$

on resolving into partial fractions. Expanding each fraction by the Binomial Theorem, and putting $x=1$ in the expansion, we see that the first n terms of the original series are equal to

$$9(1+1+1+\ldots\text{to } n \text{ terms})-8\{1-\tfrac{1}{2}+\tfrac{1}{4}-\tfrac{1}{8}+\ldots+(-\tfrac{1}{2})^{n-1}\}$$

$$=9n-8\{1-(-\tfrac{1}{2})^n\}/\{1-(-\tfrac{1}{2})\}=9n-\tfrac{16}{3}\{1-(-\tfrac{1}{2})^n\}.$$

Examples on Chapter V

Show that the series whose general terms are given below are recurring series and find their scales of relation :

1. $u_n=A+Bn+Cn^2$. 2. $u_n=3^nA+4^nB$.

3. $u_n=(A+Bn)2^n x^n$.

4. Show that the series $1^3+2^3+3^3+\ldots+n^3+\ldots$ is a recurring series.

Find the scale of relation and the generating function of the following series :

5. $1+2x+5x^2+14x^3+\ldots$. 6. $1+2x+5x^2+12x^3+\ldots$.

7. $2-x+5x^2-7x^3+\ldots$.

8. $2+5x+10x^2+17x^3+26x^4+37x^5+\ldots$. [*Utkal*, 1949]

Find the nth term of the following recurring series:

9. $3+5+7+5+\ldots$. 10. $-\frac{3}{2}+2+0+8+\ldots$.

11. $2+7x+25x^2+91x^3+\ldots$. 12. $5-2x+8x^2+4x^3+\ldots$.

Find the sum to n terms of the following recurring series:

13. $2+5x+8x^2+11x^3+\ldots$. 14. $-1+6x^2+30x^3+\ldots$.

15. $2+6+14+30+\ldots$. [*Madras*, 1949]

16. $1+2+3+5+7+9+\ldots$.

17. $2-5+29-89+\ldots$. [*Nagpur*, 1949]

18. Find the nth term of a recurring series of which the first four terms are

$$1+2x+7x^2+20x^3.$$

Find also the sum of the first n terms when $x=-1$. [*Nagpur*, 1948]

19. Find the generating function, general term, and the sum to n terms of the recurring series

$$1-7x-x^2-43x^3-\ldots. \quad [\textit{Sagar}, 1950]$$

20. The scales of the recurring series

$$a_0+a_1x+a_2x^2+a_3x^3+\ldots$$

and

$$b_0+b_1x+b_2x^2+b_3x^3+\ldots$$

are $1+px+qx^2$ and $1+rx+sx^2$ respectively. Show that the series whose general term is $(a_n+b_n)x^n$ is a recurring series whose scale is

$$1+(p+r)x+(q+s+pr)x^2+(qr+ps)x^3+qsx^4.$$

CHAPTER VI

CONTINUED FRACTIONS

6·1. Definitions. An expression of the form

$$a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \dots}},$$

where a_1, b_2, ... are any numbers, is called a *continued fraction*. For convenience it is written as

$$a_1 + \frac{b_2}{a_2 +}\,\frac{b_3}{a_3 +}\,\dots .$$

We shall consider, however, only the simpler form

$$a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3 +}\,\dots ,$$

in which $a_1, a_2, \dots$ are all positive integers, except that a_1 may be zero. Such a fraction is called a *simple* continued fraction. When the number of quotients $a_1, a_2, \dots$ is finite, the continued fraction is said to be *terminating*; otherwise it is called an *infinite* continued fraction.

The usual arithmetical process of evaluating a terminating continued fraction is to simplify the fraction step by step, proceeding from the extreme right towards the left (or from the bottom upwards). Our object in the present chapter will be to obtain approximations to the fraction, starting from the left (or the top), and to study the properties of these approximations.

We shall regard the quantities

$$a_1,\, a_1+\frac{1}{a_2},\; a_1+\frac{1}{a_2+}\frac{1}{a_3},\; \ldots,$$

as approximations to the value of the continued fraction

$$a_1+\frac{1}{a_2+}\frac{1}{a_3+}\frac{1}{a_4+}\cdots.$$

They are called the 1st, 2nd, 3rd, ... *convergents* respectively.

6·2. A Property of Convergents. *The convergents are alternately less and greater than the continued fraction.*

Let the continued fraction be

$$a_1+\frac{1}{a_2+}\frac{1}{a_3+}\frac{1}{a_4+}\cdots,$$

where, by hypothesis, a_1, a_2, ... are all *positive* integers.

Then the first convergent a_1 is less than the continued fraction, because the part

$$\frac{1}{a_2+}\frac{1}{a_3+}\cdots$$

is omitted. The second convergent a_1+1/a_2 is greater than the continued fraction, because the denominator is too small, the part $1/a_3+\ldots$ being omitted. The third convergent $a_1+1/(a_2+1/a_3)$ is less than the continued fraction, because a_2+1/a_3 is too great, a part of its denominator being omitted; and so on. Hence the proposition.

It may be noted that the convergents of odd order are less and those of even order greater than the continued fraction.

6·3. Conversion into a continued fraction. *To convert an ordinary fraction m/n into a simple continued fraction.*

Divide m by n, and let a_1 be the quotient and p the remainder; then

$$\frac{m}{n}=a_1+\frac{p}{n}=a_1+\frac{1}{n/p}.$$

Divide n by p, let a_2 be the quotient and q the remainder; then

$$\frac{n}{p}=a_2+\frac{q}{p}=a_2+\frac{1}{p/q},$$

so that
$$\frac{m}{n}=a_1+\frac{1}{a_2+}\,\frac{1}{p/q}.$$

Similarly, we can divide p by q, getting a quotient a_3 and a remainder r, and so on. Thus, we get

$$\frac{m}{n}=a_1+\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots.$$

Ex. Express 217/502 as a continued fraction.

By actual division as in the margin we see that

$$\frac{217}{502}=\frac{1}{502/217}$$

$$=\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{5+}\,\frac{1}{4+}\,\frac{1}{3}.$$

```
217)502(2
    434
    68)217(3
       204
       13)68(5
          65
          3)13(4
            12
            1)3(3
              3
              ×
```

6·4. Formation of Convergents. *To establish the law of formation of the successive convergents.*

Let the continued fraction be

$$a_1+\frac{1}{a_2+}\,\frac{1}{a_3+}\,\frac{1}{a_4+}\cdots;$$

then the successive convergents can be written as

$$\frac{a_1}{1}, \quad \frac{a_2a_1+1}{a_2}, \quad \frac{a_3(a_2a_1+1)+a_1}{a_3a_2+1}, \ldots.$$

If we denote the numerators of these convergents by $p_1, p_2, p_3, \ldots$ and the denominators by $q_1, q_2, q_3, \ldots$ respectively, we notice that

$$p_3=a_3p_2+p_1 \quad \text{and} \quad q_3=a_3q_2+q_1.$$

It may be verified that p_4 and q_4 can be expressed in a similar way.

We shall prove by mathematical induction that

$$\mathbf{p}_n=\mathbf{a}_n\mathbf{p}_{n-1}+\mathbf{p}_{n-2} \quad \text{and} \quad \mathbf{q}_n=\mathbf{a}_n\mathbf{q}_{n-1}+\mathbf{q}_{n-2}. \quad (1)$$

Suppose that these relations are true for any particular value of n. Then the nth convergent is

$$\frac{a_np_{n-1}+p_{n-2}}{a_nq_{n-1}+q_{n-2}}.$$

The $(n+1)$th convergent differs from the nth in having the quotient a_n+1/a_{n+1} in the place of a_n. Therefore the $(n+1)$th convergent is equal to

$$\frac{(a_n+1/a_{n+1})p_{n-1}+p_{n-2}}{(a_n+1/a_{n+1})q_{n-1}+q_{n-2}}=\frac{a_{n+1}(a_np_{n-1}+p_{n-2})+p_{n-1}}{a_{n+1}(a_nq_{n-1}+q_{n-2})+q_{n-1}}$$

$$=\frac{a_{n+1}p_n+p_{n-1}}{a_{n+1}q_n+q_{n-1}}, \text{ by (1).}$$

Hence if we put the numerator equal to p_{n+1} and the denominator equal to q_{n+1}, then p_{n+1} and q_{n+1} are given by the same rule as (1).

Thus, if the law (1) is true for the nth convergent, it is also true for the $(n+1)$th convergent. But we have seen that it is true for the third convergent; hence it is true for every value of n.

Ex. If p_n/q_n is the nth convergent of

$$a_1+\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots\frac{1}{a_n+}\cdots,$$

show that $\dfrac{p_n}{p_{n-1}}=a_n+\dfrac{1}{a_{n-1}+}\,\dfrac{1}{a_{n-2}+}\cdots\dfrac{1}{a_2+}\,\dfrac{1}{a_1}$,

and $\dfrac{q_n}{q_{n-1}}=a_n+\dfrac{1}{a_{n-1}+}\,\dfrac{1}{a_{n-2}+}\cdots\dfrac{1}{a_2}$. [*Gorakhpur*, '59]

Since $p_n=a_np_{n-1}+p_{n-2}$, therefore

$$\frac{p_n}{p_{n-1}}=a_n+\frac{p_{n-2}}{p_{n-1}}=a_n+\frac{1}{p_{n-1}/p_{n-2}}.$$

But $p_{n-1}=a_{n-1}p_{n-2}+p_{n-3}$; therefore

$$\frac{p_{n-1}}{p_{n-2}}=a_{n-1}+\frac{1}{p_{n-2}/p_{n-3}},$$

so that
$$\frac{p_n}{p_{n-1}}=a_n+\frac{1}{a_{n-1}+}\,\frac{1}{p_{n-2}/p_{n-3}}.$$

Proceeding in the same way, we get

$$\frac{p_n}{p_{n-1}}=a_n+\frac{1}{a_{n-1}+}\,\frac{1}{a_{n-2}+}\cdots\frac{1}{a_3+}\,\frac{1}{p_2/p_1}. \tag{1}$$

Now
$$\frac{p_1}{q_1}=\frac{a_1}{1} \text{ and } \frac{p_2}{q_2}=\frac{a_1a_2+1}{a_2}. \tag{2}$$

Therefore $p_2/p_1=(a_1a_2+1)/a_1=a_2+1/a_1$. Substituting this in (1), we obtain the first result.

Starting from $q_n=a_nq_{n-1}+q_{n-2}$, and proceeding as above, we get

$$\frac{q_n}{q_{n-1}}=a_n+\frac{1}{a_{n-1}+}\cdots\frac{1}{a_3+}\,\frac{1}{q_2/q_1}$$
$$=a_n+\frac{1}{a_{n-1}+}\cdots\frac{1}{a_3+}\,\frac{1}{a_3}, \text{ by (2)}.$$

6·5. Relation between successive convergents. *If p_n/q_n be the nth convergent of a continued fraction, then*

$$p_nq_{n-1}-q_np_{n-1}=(-1)^n.$$

Let the continued fraction be

$$a_1+\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots;$$

then $$\begin{aligned} p_nq_{n-1}-q_np_{n-1} &= (a_np_{n-1}+p_{n-2})q_{n-1}-(a_nq_{n-1}+q_{n-2})p_{n-1} \\ &= (-1)(p_{n-1}q_{n-2}-q_{n-1}p_{n-2}) \\ &= (-1)^2(p_{n-2}q_{n-3}-q_{n-2}p_{n-3}), \end{aligned}$$
similarly; and a repeated application gives
$$p_nq_{n-1}-q_np_{n-1}=(-1)^{n-2}(p_2q_1-q_2p_1).$$
But $p_1=a_1$, $q_1=1$, $p_2=a_1a_2+1$, $q_2=a_2$, so that
$$p_2q_1-q_2p_1=(a_1a_2+1)-a_1a_2=1=(-1)^2.$$
Therefore
$$\mathbf{p_nq_{n-1}-q_np_{n-1}=(-1)^n.}$$
This formula will also hold for continued fractions in which a_1 is zero, provided that $1/a_2$ is reckoned as the *second* convergent.

COR. 1. p_n and q_n cannot have a common factor; for, if they have one, it will divide $p_nq_{n-1}-q_np_{n-1}$, that is, unity, which is not possible. Therefore *every convergent of a simple continued fraction is in its lowest terms.*

COR. 2. The difference between the nth and the $(n-1)$th convergent is
$$\frac{p_n}{q_n}\sim\frac{p_{n-1}}{q_{n-1}}=\frac{p_nq_{n-1}\sim q_np_{n-1}}{q_nq_{n-1}}=\frac{1}{q_nq_{n-1}}.$$

Ex. Show that $p_nq_{n-3}-q_np_{n-3}=(-1)^{n-2}(a_na_{n-1}+1)$.

We see that
$$\begin{aligned} p_nq_{n-3}-q_np_{n-3} &= (a_np_{n-1}+p_{n-2})q_{n-3}-(a_nq_{n-1}+q_{n-2})p_{n-3} \\ &= a_n(p_{n-1}q_{n-3}-q_{n-1}p_{n-3})+(p_{n-2}q_{n-3}-q_{n-2}p_{n-3}) \\ &= a_n\{(a_{n-1}p_{n-2}+p_{n-3})q_{n-3} \\ &\qquad -(a_{n-1}q_{n-2}+q_{n-3})p_{n-3}\}+(-1)^{n-2} \\ &= a_na_{n-1}(-1)^{n-2}+(-1)^{n-2}=(-1)^{n-2}(a_na_{n-1}+1). \end{aligned}$$

EXAMPLES

Convert the following fractions into continued fractions:

1. 798/383.
2. 427/2166.
3. 3·743.
4. 0·3118.

Calculate the successive convergents of

5. $2+\dfrac{1}{5+}\,\dfrac{1}{1+}\,\dfrac{1}{2+}\,\dfrac{1}{9+}\,\dfrac{1}{3}$.

6. $\dfrac{1}{2+}\,\dfrac{1}{2+}\,\dfrac{1}{3+}\,\dfrac{1}{1+}\,\dfrac{1}{2+}\,\dfrac{1}{3+}\,\dfrac{1}{2}$.

7. A metre is equal to 39·37079 inches. Show by the theory of continued fractions that 32 metres are nearly equal to 35 yards. [*Agra*, 1957]

8. A mile is equal to 1·60934 kilometres. Show that 23 miles are nearly equal to 37 kilometres.

9. Express $(2a^2-a-1)/(2a^2-3a)$ as a continued fraction.

10. Express $\dfrac{a^3+6a^2+13a+10}{a^4+6a^3+14a^2+15a+7}$ as a continued fraction, and find the third convergent.

11. Show that $\dfrac{p_{n+1}-p_{n-1}}{q_{n+1}-q_{n-1}}=\dfrac{p_n}{q_n}$.

12. If p_n/q_n is the nth convergent of a continued fraction, and a_n the corresponding quotient, show that

$$p_{n+2}q_{n-2}\sim p_{n-2}q_{n+2}=a_{n+2}a_{n+1}a_n+a_{n+2}+a_n.$$

6·6. Complete Quotient. Let p_n/q_n be the nth convergent of the continued fraction

$$a_1+\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots\frac{1}{a_n+}\,\frac{1}{a_{n+1}+}\cdots; \qquad (1)$$

then $$\frac{p_n}{q_n}=a_1+\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots\frac{1}{a_n}.$$

The continued fraction (1) differs from p_n/q_n in having

$$a_n+\frac{1}{a_{n+1}+}\,\frac{1}{a_{n+2}+}\cdots \qquad (2)$$

in the place of a_n. For this reason a_n is generally called the nth *partial* quotient, while (2) is called the nth *complete* quotient.

Since $$\frac{p_n}{q_n}=\frac{a_np_{n-1}+p_{n-2}}{a_nq_{n-1}+q_{n-2}},$$

we can obtain the value of the continued fraction by replacing a_n by the complete quotient. Thus if x denotes the value of the continued fraction and k that of the complete quotient, we get

$$x=\frac{kp_{n-1}+p_{n-2}}{kq_{n-1}+q_{n-2}}.$$

6·61. A Property of Convergents. *Each convergent is a closer approximation to the value of the continued fraction than the preceding convergent.*

Let x be the value of the continued fraction, and p_n/q_n and p_{n+1}/q_{n+1} two successive convergents. Then, if k is the $(n+2)$th *complete* quotient, we have

$$x=\frac{kp_{n+1}+p_n}{kq_{n+1}+q_n}.$$

Therefore $$x\sim\frac{p_n}{q_n}=\frac{(kp_{n+1}+p_n)q_n\sim(kq_{n+1}+q_n)p_n}{(kq_{n+1}+q_n)q_n}$$

$$=\frac{k}{(kq_{n+1}+q_n)q_n}, \qquad (1)$$

and $$x\sim\frac{p_{n+1}}{q_{n+1}}=\frac{(kp_{n+1}+p_n)q_{n+1}\sim(kq_{n+1}+q_n)p_{n+1}}{(kq_{n+1}+q_n)q_{n+1}}$$

$$=\frac{1}{(kq_{n+1}+q_n)q_{n+1}}. \qquad (2)$$

Since $k>1$ and $q_n<q_{n+1}$, (1) is greater than (2). Therefore, p_{n+1}/q_{n+1} is a closer approximation to x than the preceding convergent p_n/q_n.

COR. 1. *Each convergent is a closer approximation to the value of the continued fraction than* ANY *of the preceding convergents.*

COR. 2. Combining the result of the above article with that of § 6·2, we see that *the convergents of odd order are all less than the continued fraction, but increase steadily, while those of even order are all greater than the continued fraction, but decrease steadily.* Thus the value of the continued fraction lies between any two successive convergents.

6·62. Limits to Error. *To find limits to the error made in taking any convergent for the continued fraction.*

Let p_n/q_n be the nth convergent and x the value of the continued fraction; then the numerical error in taking p_n/q_n instead of x is $x \sim p_n/q_n$.

If k is the $(n+2)$th complete quotient, we have

$$x=\frac{kp_{n+1}+p_n}{kq_{n+1}+q_n}.$$

Therefore the error, i.e., $x \sim p_n/q_n$

$$=\frac{(kp_{n+1}+p_n)q_n \sim (kq_{n+1}+q_n)p_n}{(kq_{n+1}+q_n)q_n}=\frac{k}{(kq_{n+1}+q_n)q_n}$$

$$=\frac{1}{q_n(q_{n+1}+q_n/k)}. \qquad \ldots \qquad (1)$$

Since $k>1$, so $q_n/k<q_n$ and the error is greater than

$$\frac{1}{q_n(q_{n+1}+q_n)}. \qquad \ldots \qquad (2)$$

Also, by (1), the error is less than

$$\frac{1}{q_nq_{n+1}}. \qquad \ldots \qquad (3)$$

Corollary. We find by (3) that the error is less than

$$1/q_n(a_{n+1}q_n+q_{n-1}).$$

or, less than $$\mathbf{1/a_{n+1}q_n{}^2}. \tag{4}$$

This is small when a_{n+1} is large. Therefore, *if any of the quotients is very large, the convergent just preceding it gives a fairly close approximation to the continued fraction.*

Of course, taking even one more quotient will give a better approximation (§ 6·61), but the labour involved will be out of proportion to the extra accuracy gained.

From (4) we also see that the error is less than $1/q_n{}^2$. This can be taken as a rough upper limit to the error.

Ex. Find a good approximation to the value of π, given that

$$\pi=3+\frac{1}{7+}\;\frac{1}{15+}\;\frac{1}{1+}\;\frac{1}{292+}\;\frac{1}{1+}\;\frac{1}{1+}\cdots$$

The fourth convergent, which precedes the large quotient 292, would give an easily calculated good approximation. Its value is 355/113, and the error is less than

$$1/292\times 113^2, \text{ i.e., } 0{\cdot}000{,}0003.$$

6·63. Another Property of Convergents. *A convergent is nearer to the continued fraction than any other fraction whose denominator is less than that of the convergent.*

Let p_n/q_n be any convergent of the continued fraction x, and let r/s be a fraction whose denominator is less than q_n.

We have to show that p_n/q_n is nearer to x than is r/s.

Proof. If possible, suppose that r/s is nearer to x than p_n/q_n. Then r/s is nearer to x than the preceding convergent p_{n-1}/q_{n-1} also (by § 6·61).

Now the convergents are alternately less and greater than the continued fraction, so that x lies between p_{n-1}/q_{n-1} and p_n/q_n. Therefore r/s, which by hypothesis is nearer to x than either p_{n-1}/q_{n-1} or p_n/q_n, also lies between p_{n-1}/q_{n-1} and p_n/q_n.

Hence $$\frac{r}{s}\sim\frac{p_{n-1}}{q_{n-1}}<\frac{p_n}{q_n}\sim\frac{p_{n-1}}{q_{n-1}}, \text{ i.e., } <\frac{1}{q_nq_{n-1}}.$$

Multiplying the first and the last members by sq_{n-1}, we get

$$rq_{n-1} \sim sp_{n-1} < s/q_n,$$

which shows that an integer is less than a proper fraction. As this is not possible, p_n/q_n must be nearer to x than r/s.

Ex. If p/q and p'/q' be any two consecutive convergents of a continued fraction x, show that

$$\frac{pp'}{qq'} \substack{<\\>} x^2 \text{ according as } \frac{p}{q} \substack{>\\<} \frac{p'}{q'}. \quad [\textit{Banaras}, 1960]$$

Let k be the complete quotient corresponding to the convergent next after p'/q'; then

$$x = \frac{kp'+p}{kq'+q}.$$

Therefore $$\frac{pp'}{qq'} - x^2 = \frac{pp'(kq'+q)^2 - qq'(kp'+p)^2}{qq'(kq'+q)^2}$$

$$= \frac{k^2p'q'(pq'-qp') + pq(p'q-q'p)}{qq'(kq'+q)^2}$$

$$= \frac{(k^2p'q'-pq)(pq'-qp')}{qq'(kq'+q)^2}.$$

Since $k>1$, $p'>p$ and $q'>q$, therefore $k^2p'q'-pq$ is positive. Hence $pp'/qq'-x^2$ is positive or negative according as

$$pq'-p'q \substack{>\\<} 0 \quad \text{or} \quad \frac{p}{q} \substack{>\\<} \frac{p'}{q'}.$$

Examples

1. A kilometre is equal to 0·62137 miles, nearly. Show that 5/8, 18/29, 23/27, 64/103 are successive approximations to the ratio of a kilometre to a mile, and find limits to the error made in taking the first and the last values.

2. If $$\sqrt{11} = 3 + \frac{1}{3+}\,\frac{1}{6+}\,\frac{1}{3+}\,\frac{1}{6+}\cdots,$$

leow that the error made in taking the fourth convergent is sssh than 0·00005.

3. Find an approximation to

$$1+\frac{1}{3+}\frac{1}{5+}\frac{1}{7+}\frac{1}{9+}\frac{1}{11+}\cdots$$

which differs from the true value by less than 0·0001.

[*Allahabad*, 1960]

4. Given that 1 kilogramme$=2{\cdot}2046$ pounds, show that a weight of 44 kilogrammes is slightly greater than 97 pounds, the error per kilogramme being less than half a grain.

5. Find the convergent, the denominator of which does not exceed 1000, which is the best representative of the continued fraction

$$\frac{1}{1+}\frac{1}{2+}\frac{1}{3+}\frac{1}{4+}\cdots$$

and show that the error in adopting this value is less than $\frac{1}{2}\times 10^{-5}$ and greater than $\frac{1}{4}\times 10^{-5}$. [*U.P.C.S.*, 1960]

6. Show that the error made in taking the nth convergent p_n/q_n for the continued fraction lies between

$$a_{n+2}/q_n q_{n+2} \quad \text{and} \quad (a_{n+2}+1)/q_n q_{n+2}.$$

6·7. Recurring continued fractions. If in an infinite continued fraction a fixed number of quotients are, after some stage, repeated again and again, the fraction is known as a *recurring* continued fraction.

A simple example of a recurring continued fraction is

$$a+\frac{1}{a+}\frac{1}{a+}\frac{1}{a+}\cdots, \tag{1}$$

in which the quotient a is repeated again and again.

If p_n/q_n is the nth convergent of (1), then

$$p_n=ap_{n-1}+p_{n-2} \quad \text{and} \quad q_n=aq_{n-1}+q_{n-2},$$

for every value of n greater than 2. These equations give a scale of relation, showing that Σp_n and Σq_n are two recurring series. This property can be used to determine the nth convergent of (1), as in Ex. 2 below.

To find the value of the continued fraction (1), put it equal to x; then

$$x=a+1/x.$$

This gives $$x^2-ax-1=0,$$

or $$x=\tfrac{1}{2}\{a+\sqrt{(a^2+4)}\},$$

the negative root being inadmissible.

A similar method can be applied to find the value of any recurring continued fraction.

The student should note that infinite processes do not always have a meaning (see § 7·1). But in the case of an infinite simple continued fraction, the successive convergents continually approach a definite value, which can be regarded as the value of the continued fraction.

Ex. 1. Find the value of $\dfrac{1}{1+}\,\dfrac{1}{3+}\,\dfrac{1}{1+}\,\dfrac{1}{3+}\cdots$.

Let the value of the continued fraction be x. Then

$$x=\frac{1}{1+}\,\frac{1}{3+x}=\frac{3+x}{4+x},$$

or $$x^2+3x-3=0.$$

Therefore $$x=\tfrac{1}{2}(-3+\sqrt{21}),$$

the negative root being inadmissible.

Ex. 2. Find the nth convergent of $1+\dfrac{1}{2+}\,\dfrac{1}{2+}\,\dfrac{1}{2+}\cdots$.

Let p_n/q_n be the nth convergent, and consider the two series $\Sigma p_n x^n$ and $\Sigma q_n x^n$.

Since the successive convergents of the given continued fraction are

$$\tfrac{1}{1},\ \tfrac{3}{2},\ \tfrac{7}{5},\ \tfrac{17}{12},\ \ldots, \qquad (1)$$

therefore $\Sigma p_n x^n = x+3x^2+7x^3+17x^4+\ldots+p_n x^n+\ldots,$ (2)

and $\Sigma q_n x^n = x+2x^2+5x^3+12x^4+\ldots+q_n x^n+\ldots.$ (3)

Series (2) is a recurring series with the scale of relation

$$p_n - 2p_{n-1} - p_{n-2} = 0.$$

Therefore, if S is the generating function of (2), we have

$$\begin{aligned} S &= x+3x^2+7x^3+17x^4+\ldots, \\ -2xS &= \quad -2x^2-6x^3-14x^4-\ldots, \\ \text{and} \quad -x^2S &= \qquad\qquad -x^3-3x^4-\ldots, \end{aligned}$$

whence, on addition,

$$S = \frac{x+x^2}{1-2x-x^2}.$$

Resolving this into partial fractions, we get

$$\begin{aligned} S &= -1 - \frac{\frac{1}{2}(\sqrt{2}-1)}{x-(\sqrt{2}-1)} + \frac{\frac{1}{2}(\sqrt{2}+1)}{x+(\sqrt{2}+1)} \\ &= -1 - \frac{\frac{1}{2}}{(\sqrt{2}+1)x-1} + \frac{\frac{1}{2}}{(\sqrt{2}-1)x+1} \\ &= -1 + \tfrac{1}{2}\{1-(\sqrt{2}+1)x\}^{-1} + \tfrac{1}{2}\{1+(\sqrt{2}-1)x\}^{-1}. \end{aligned}$$

This on expansion generates (2). Therefore p_n, i.e., the coefficient of x^n, is

$$\tfrac{1}{2}(\sqrt{2}+1)^n + \tfrac{1}{2}(-1)^n(\sqrt{2}-1)^n.$$

Proceeding similarly, we find that the generating function of series (3) is

$$\begin{aligned} \frac{x}{1-2x-x^2} &= -\frac{1}{2\sqrt{2}}\left\{\frac{\sqrt{2}-1}{x-(\sqrt{2}-1)} + \frac{\sqrt{2}+1}{x+(\sqrt{2}+1)}\right\} \\ &= (1/2\sqrt{2})[\{1-(\sqrt{2}+1)x\}^{-1} - \{1+(\sqrt{2}-1)x\}^{-1}]. \quad (4) \end{aligned}$$

Therefore q_n, the coefficient of x^n in the expansion of (4)

$$= (1/2\sqrt{2})[(\sqrt{2}+1)^n - (-1)^n(\sqrt{2}-1)^n].$$

Hence the nth convergent p_n/q_n

$$= \frac{\sqrt{2}\{(\sqrt{2}+1)^n+(-1)^n(\sqrt{2}-1)^n\}}{(\sqrt{2}+1)^n-(-1)^n(\sqrt{2}-1)^n}.$$

6·8. Conversion of a Quadratic Surd. Quadratic surds and irrational numbers can be easily converted into continued fractions by the method given below, which is essentially the same as that of § 6·3.

Let x be the given number, and let

$$x = a_1 + 1/b_1,$$

where a_1 is the integral part of x and $1/b_1$ is the remainder. Evidently, $1/b_1 < 1$, so that $b_1 > 1$. Let

$$b_1 = a_2 + 1/b_2,$$

where a_2 is the integral part of b_1 and $1/b_2$ is the remainder. Again, b_2 can be broken up into the integral part a_3 and the remainder $1/b_3$; and so on. Thus

$$x = a_1 + \frac{1}{b_1} = a_1 + \frac{1}{a_2+}\,\frac{1}{b_2} = a_1 + \frac{1}{a_2+}\,\frac{1}{a_3+}\cdots.$$

Ex. Express $\sqrt{7}$ as a simple continued fraction. [*Agra*, '55]

Since $2 < \sqrt{7} < 3$, the integral part of $\sqrt{7}$ is 2. Hence we have

$$\sqrt{7} = 2 + (\sqrt{7} - 2) = 2 + \frac{3}{\sqrt{7}+2}, \qquad (1)$$

$$\frac{\sqrt{7}+2}{3} = 1 + \frac{\sqrt{7}-1}{3} = 1 + \frac{6}{3(\sqrt{7}+1)} = 1 + \frac{2}{\sqrt{7}+1}, \qquad (2)$$

$$\frac{\sqrt{7}+1}{2} = 1 + \frac{\sqrt{7}-1}{2} = 1 + \frac{6}{2(\sqrt{7}+1)} = 1 + \frac{3}{\sqrt{7}+1}, \qquad (3)$$

$$\frac{\sqrt{7}+1}{3} = 1 + \frac{\sqrt{7}-2}{3} = 1 + \frac{3}{3(\sqrt{7}+2)} = 1 + \frac{1}{\sqrt{7}+2}, \qquad (4)$$

$$\sqrt{7}+2 = 4 + (\sqrt{7}-2) = 4 + \frac{3}{\sqrt{7}+2}; \qquad (5)$$

after which the steps (2) to (5) repeat themselves. Therefore

$$\sqrt{7} = 2 + \frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{4+}\cdots,$$

the last four quotients being repeated again and again.

6·9. General Continued Fractions. For the continued fraction

$$a_1+\frac{b_2}{a_2+}\,\frac{b_3}{a_3+}\,\frac{b_4}{a_4+}\cdots,$$

the successive convergents are defined to be

$$a_1,\ a_1+\frac{b_2}{a_2},\ a_1+\frac{b_2}{a_2+}\,\frac{b_3}{a_3},\ \ldots.$$

The law of formation of the nth convergent p_n/q_n is

$$p_n=a_np_{n-1}+b_np_{n-2},\ q_n=a_nq_{n-1}+b_nq_{n-2}. \tag{1}$$

The proof is similar to that of § 6·4. But, for the general continued fraction, p_n/q_n is not necessarily in its lowest terms.

Most of the problems on general continued fractions can be solved with the help of relations (1), but sometimes it is simpler to proceed from first principles, as in the example below.

Ex. Find the nth convergent to

$$\frac{1}{2-}\,\frac{1}{2-}\,\frac{1}{2-}\cdots. \qquad [\textit{Banaras}, 1949]$$

Let p_n/q_n denote the nth convergent; then

$$\frac{p_1}{q_1}=\frac{1}{2},\ \frac{p_2}{q_2}=\frac{2}{3},\ \frac{p_3}{q_3}=\frac{3}{4},\ \ldots.$$

We notice that for the first three convergents

$$p_n/q_n=n/(n+1). \quad . \quad . \quad . \quad (1)$$

Suppose this is true up to a particular value of n; then

$$\frac{p_{n+1}}{q_{n+1}}=\frac{1}{2-}\,\frac{1}{2-}\,\frac{1}{2-}\ldots\text{to } (n+1) \text{ quotients}=\frac{1}{2-}\,\frac{p_n}{q_n}$$

$$=\frac{1}{2-n/(n+1)}=\frac{n+1}{n+2}.$$

This is of the same form as (1). Therefore, if (1) is true for the nth convergent, it is true for the $(n+1)$th convergent also. But we have seen that (1) is true for the third convergent. Hence it is true for all the succeeding convergents.

EXAMPLES

Find the value of the following recurring continued fractions :

1. $2+\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{4+}\,\frac{1}{4+}\cdots.$

2. $1+\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{3+}\cdots.$ [*Agra*, 1946]

3. $\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{4+}\cdots.$

4. Prove that, if each fraction contains n elements,

$$\frac{2}{4+}\,\frac{2}{4+}\,\frac{2}{4+}\,\frac{2}{4+}\cdots=\frac{1}{2+}\,\frac{1}{4+}\,\frac{1}{2+}\,\frac{1}{4+}\cdots.$$

5. If

$$x=a+\frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{a+}\cdots \text{ and } y=b+\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{b+}\cdots$$

prove that $bx=ay$. [*Allahabad*, 1957]

Convert the following surds into continued fractions :

6. $\sqrt{5}$.
7. $\sqrt{6}$.
8. $\sqrt{10}$. [*Agra*, 1949]
9. $\sqrt{19}$.

10. Show that $\sqrt{(a^2+1)}=a+\frac{1}{2a+}\,\frac{1}{2a+}\,\frac{1}{2a+}\cdots.$

11. Prove that the nth convergent of $2+\frac{1}{2+}\,\frac{1}{2+}\cdots$ is

$$\{(1+\sqrt{2})^{n+1}-(1-\sqrt{2})^{n+1}\}/\{(1+\sqrt{2})^n-(1-\sqrt{2})^n\}.$$

12. Find the nth convergent of $\frac{3}{2+}\,\frac{3}{2+}\,\frac{3}{2+}\cdots.$

EXAMPLES ON CHAPTER VI

1. Find the value of

$$\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{1+}\cdots.$$ [*Rajputana*, '50]

2. Two scales of equal length are divided into 162 and 209 equal parts respectively; if their zero points be coincident, show that the 31st division of one and the 40th division of the other are nearest. [*Nagpur*, 1954]

3. If $(n^4+n^2-1)/(n^3+n^2+n+1)$ is converted into a continued fraction, show that the quotients are $n-1$ and $n+1$ alternately, and find the successive convergents. [*Sagar*, '48]

4. Show that

$$a\left(x_1+\frac{1}{ax_2+}\,\frac{1}{x_3+}\,\frac{1}{ax_4+}\ldots\text{ to } 2n \text{ quotients}\right)$$
$$=ax_1+\frac{1}{x_2+}\,\frac{1}{ax_3+}\,\frac{1}{x_4+}\ldots\text{to } 2n \text{ quotients.}$$

5. If M/N, P/Q, R/S are the nth, $(n-1)$th, $(n-2)$th convergents of the continued fractions

$$\frac{1}{a_1+}\,\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots,\quad \frac{1}{a_2+}\,\frac{1}{a_3+}\,\frac{1}{a_4+}\cdots,\quad \frac{1}{a_3+}\,\frac{1}{a_4+}\,\frac{1}{a_5+}\cdots,$$

respectively, show that

$$M=a_2P+R,\quad N=(a_1a_2+1)P+a_1R.$$

6. Show that the difference between the first and the nth convergents is numerically equal to

$$\frac{1}{q_1q_2}-\frac{1}{q_2q_3}+\frac{1}{q_3q_4}-\cdots+\frac{(-1)^n}{q_{n-1}q_n}.$$

[*Rajasthan*, 1960]

7. If p_n/q_n is the nth convergent of

$$\frac{1}{a+}\,\frac{1}{a+}\,\frac{1}{a+}\cdots,$$

show that

(i) $p_n^2+p_{n+1}^2=p_{n-1}p_{n+1}+p_np_{n+2}$,

(ii) $p_n=q_{n-1}$. [*Sagar*, 1950]

8. If p_n/q_n is the nth convergent of

$$\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{b+}\cdots,$$

show that

$$p_{n+2}-(ab+2)p_n+p_{n-2}=0,\quad q_{n+2}-(ab+2)q_n+q_{n-2}=0.$$

[*Agra*, 1954]

9. If p_n/q_n is the nth convergent of the continued fraction

$$\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{b+}\cdots,$$

show that
$$q_{2n}=p_{2n+1},\ q_{2n-1}=(a/b)p_{2n}.$$

10. If p_n/q_n is the nth convergent of the continued fraction

$$\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{c+}\,\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{c+}\cdots,$$

show that
$$p_{3n+3}=bp_{3n}+(bc+1)q_{3n}.$$

11. If p_n/q_n and p_{n-1}/q_{n-1} be the last and the last but one convergents of

$$\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{c+}\cdots\frac{1}{k+}\,\frac{1}{l},$$

show that

$$\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{c+}\cdots\frac{1}{k+}\,\frac{1}{l+}\,\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{c+}\cdots\frac{1}{k+}\,\frac{1}{l}=\frac{p_nq_n+pp_{n-1}}{q_n{}^2+p_nq_{n-1}}.$$

[*Nagpur*, 1949]

12. If p_n/q_n is the nth convergent of

$$\frac{1}{a+}\,\frac{1}{a+}\,\frac{1}{a+}\cdots,$$

show that p_n and q_n are respectively the coefficients of x^n in the expansions of

$$\frac{x}{1-ax-x^2}\quad\text{and}\quad\frac{ax+x^2}{1-ax-x^2}.$$

Hence show that $p_n=q_{n-1}=(\alpha^n-\beta^n)/(\alpha-\beta)$ where α, β are the roots of the equation $t^2-at-1=0$.

[*Allahabad*, 1952]

CHAPTER VII

CONVERGENCE OF SERIES

7·1. Infinite Series. Let u_n be a function of n, which has a definite value for all positive integral values of n. Then an expression of the form

$$u_1+u_2+u_3+\ldots+u_n+\ldots,$$

in which every term is followed by another term, is called an *infinite series*. The above series is denoted by

$$\sum_{n=1}^{\infty} u_n \text{ or } \Sigma u_n,$$

and the sum of its first n terms is denoted by s_n.

An example of an infinite series is the geometrical series

$$1+\tfrac{1}{2}+(\tfrac{1}{2})^2+(\tfrac{1}{2})^3+\ldots. \tag{1}$$

Let us consider the sum of its first n terms. For this series

$$s_n=2-(\tfrac{1}{2})^{n-1}, \text{ or } 2-s_n=(\tfrac{1}{2})^{n-1}.$$

We see that when n gets larger and larger, $(\frac{1}{2})^{n-1}$ becomes smaller and smaller. By choosing n sufficiently large, $(\frac{1}{2})^{n-1}$ can be made as small as we please; and thus $2-s_n$ can be made as small as we please. We express this fact by saying that s_n approaches the limit 2; or, that the limit of s_n, as n tends to infinity, is 2. A series like (1), for which s_n tends to a finite limit, is called a convergent series.

Now consider the infinite series

$$1+2+2^2+2^3+\ldots. \tag{2}$$

In this case $s_n=2^n-1$. As n increases, s_n also increases; and by choosing n sufficiently large, s_n can be made to exceed any

given number, however large that number may be. We express this by saying that s_n tends to infinity as n tends to infinity. A series like (2) is called a divergent series.

There is a third type of series: consider for example

$$1-1+1-1+1-1+\dots,$$

or

$$1-3+5-7+9-11+\dots.$$

In the first series s_n is equal to 1 or 0, according as n is odd or even. In the second series s_n is alternately positive and negative, while numerically it goes on increasing as n increases. Such series, in which the sum to n terms fluctuates from one value to another, or from a positive value to a negative value, and does not approach a limit, are called oscillatory series.

Rules of algebra, like addition, multiplication, etc. cannot be always applied to infinite series. For example, denoting the series (2) by S, and writing

$$S=1+2+4+8+16+\dots,$$

we have

$$2S=\quad 2+4+8+16+\dots.$$

By subtraction we get $S=-1$, which is absurd, since all the terms in S are positive. It can be shown that the addition of two infinite series is valid for convergent series only; and their multiplication is valid for a particular class of convergent series. For this reason, it is of importance to find out whether a given series is convergent or not.

7·2. Limits. In the illustrations above we have introduced the concept of limits. But we must define precisely what we mean when we say that s_n tends to a finite limit as n tends to infinity.

In an infinite series the general term u_n is a function of n. Therefore s_n, the sum of the first n terms, is also a function of n. When n varies, s_n will also vary. Now suppose that n takes successively larger and larger values. It may then happen that the corresponding values of s_n continually approach some definite number s. In this case we say that s_n tends to the limit s as n tends to infinity. But this definition is not very satisfactory. Firstly, we are using a geometrical

notion when we say that s_n approaches s. This can be avoided by saying that the difference between s_n and s becomes smaller and smaller. Since we are concerned with the numerical smallness of the difference, we say that $|s_n - s|$, i.e., $s_n - s$ taken positively, becomes smaller and smaller.

But how small should $|s_n - s|$ become ? For the series (1) of the preceding article $|s_n - s|$ is 0·063 for $n=5$, 0·002 for $n=10$, 0·00006 for $n=15$, and so on. It is clear that by taking n sufficiently large, $|s_n - s|$ can be made as small as we please. For example, by a suitable choice of n $|s_n - s|$ can be made less than 1/1000; it can be made even less than 1/1,000,000; in fact, it can be made less than any positive number ϵ, however small ϵ may be.

Therefore we can give the following definition. If having chosen any positive number ϵ, however small, we have

$$|s_n - s| < \epsilon$$

for all sufficiently large values of n, s_n is said to have the limit s as n tends to infinity.

This definition is satisfactory except for one point. It has not been made clear how large n should be. This can be done as follows. Consider again the series (1) of the preceding article. For this series $|s_n - s| = (\frac{1}{2})^{n-1}$. Thus $|s_n - s| = \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \ldots$ when $n=5, 6, 7, 8, \ldots$, respectively. Therefore if we choose ϵ as $\frac{1}{10}$,

$$|s_n - s| < \epsilon$$

when n is 5 or greater than 5. If we choose any other value of ϵ, say $\frac{1}{100}$, we can again find a number, 8 in this case, such that

$$|s_n - s| < \tfrac{1}{100} \text{ for } n \geqslant 8.$$

Similarly for any other convergent series, when the arbitrary positive number ϵ has been chosen, we can find a positive integer N such that $|s_n - s|$ is less than ϵ for all values of n greater than or equal to N. Therefore we have the following precise definition of the limit of s_n which is a function of n.

If, having chosen any positive number ϵ, however small, we can find a corresponding positive integer N, such that

$$|s_n - s| < \epsilon \text{ for } n \geqslant N,$$

s_n is said to have the limit s, as n tends to infinity.

7·21. Remarks. (i) In the above definition, N depends upon the value chosen for ϵ. This is clear from the example which has been given. The smaller the value chosen for ϵ, the larger will be the corresponding value of N.

But N will always be finite. Therefore the inequality

$$|s_n - s| < \epsilon,$$

is satisfied for all except a finite number of values of n.

(ii) The inequality

$$|s_n - s| < \epsilon,$$

shows that* $\quad s_n - s < \epsilon$ and also $s - s_n < \epsilon$,

i.e., $\quad s_n < s + \epsilon$ and $s_n > s - \epsilon$. (1)

Therefore, putting $s - \epsilon = a$ in (1), we see that, *for all except a finite number of values of n,*

$$s_n > a,$$

where a is number less than s.

Similarly, putting $s + \epsilon = b$ in (1), we see that, *for all except a finite number of values of n,*

$$s_n < b,$$

where b is any number greater than s.

(iii) We use the notation

$$\lim_{n \to \infty} s_n = s, \text{ or briefly, } \lim s_n = s,$$

to denote that the limit, as n tends to infinity, of s_n is s. We also say that s_n *tends* to s as n tends to infinity, and write $s_n \to s$ as $n \to \infty$.

7·22. Infinite Limits. It may happen in some cases [e.g., in the series (2) of § 7·1] that s_n may go on increasing as n increases; and by taking n sufficiently large, s_n can be

*Of the two quantities $s_n - s$ and $s - s_n$ one will, of course, be negative.

made to exceed any positive number A, however large A may be. In such cases, we say that s_n tends to infinity as n tends to infinity. A precise definition can be given as follows.

If, having chosen any positive number A, however large, we can find a corresponding integer N, such that

$$s_n > A \text{ for } n \geqslant N,$$

we say that s_n tends to infinity as n tends to infinity.

Similarly, s_n tends to minus infinity if s_n is negative and can be made less than any negative number $-A$, however large A may be, by taking n sufficiently large.

7·23. Evaluation of Limits. Just as we have defined the limit of s_n, similarly we can define the limit, as n tends to infinity, of any function of n. Such limits are often easy to find. Thus

$$\lim_{n\to\infty}(1/n)=0,$$

since $(1/n)-0$ can be made as small as we please by taking n sufficiently large.

We shall frequently employ the following propositions on limits which may be taken as axioms. In what follows, $\lim f(n)$ always denotes the limit, as n tends to infinity, of $f(n)$.

If $\lim f_1(n)=a$ and $\lim f_2(n)=b$, then

(i) $\lim \{f_1(n)+f_2(n)\}=a+b$,

(ii) $\lim \{f_1(n)-f_2(n)\}=a-b$,

(iii) $\lim \{k.f_1(n)\}=ka$, where k is a constant (i.e., does not depend upon n),

(iv) $\lim \{f_1(n) . f_2(n)\}=ab$,

(v) $\lim \{f_1(n)/f_2(n)\}=a/b$, provided that $b\neq 0$.

(vi) If $f_1(n)<f_2(n)$, $\lim f_1(n)\leqslant \lim f_2(n)$.

When $f_1(n)$ is less than $f_2(n)$ for all values of n, the student thinks that $\lim f_1(n)$ should be less than $\lim f_2(n)$. This may be so, but $\lim f_1(n)$ can also be equal to $\lim f_2(n)$.

For example, $1+1/n<1+2/n$, but

$$\lim(1+1/n)=\lim(1+2/n),$$

both being equal to unity.

We evaluate below some well-known limits.

(1) If p is positive, n^p tends to infinity and $1/n^p$ tends to zero as $n\to\infty$. This follows from the definition.

(2) If $x<1$, x^n tends to zero as $n\to\infty$.

If $x>1$, x^n tends to infinity as $n\to\infty$.

(3) $\lim(\log n)/n$. Put $\log n=t$ then $t\to\infty$ as $n\to\infty$. Also

$$\frac{\log n}{n}=\frac{t}{e^t}=\frac{t}{1+t+\frac{1}{2}t^2+(1/3!)t^3+\ldots}$$

$$=\frac{1}{(1/t)+1+\frac{1}{2}t+(1/3!)t^2+\ldots},$$

which can be made as small as we please by taking t sufficiently large. Therefore

$$\mathbf{lim}_{n\to\infty}\{(\log \mathbf{n})/\mathbf{n}\}=\mathbf{0}.$$

(4) To evaluate $\lim\{f_1(n)/f_2(n)\}$ when $f_1(n)$ and $f_2(n)$ are both algebraic functions of n which tend to infinity as $n\to\infty$, the numerator and denominator should first be divided by a suitable power of n, as in the example below.

Ex. 1. Find $\lim\dfrac{n^{1/3}(n^2+1)^{1/3}}{\sqrt{(2n^2+3n+1)}}$.

Dividing the numerator and the denominator by n, we get

$$\lim\frac{n^{1/3}(n^2+1)^{1/3}}{\sqrt{(2n^2+3n+1)}}=\lim\frac{(1+1/n^2)^{1/3}}{\sqrt{\{2+(3/n)+(1/n^2)\}}}=\frac{1}{\sqrt{2}},$$

since $3/n$ and $1/n^2$ both tend to zero as n tends to infinity.

Ex. 2. If $x<1$, show that $\lim nx^n=0$.

Since $x<1$, so $(1/x)>1$. Therefore $(1/x)^n$ tends to infinity. Now put $(1/x)^n=t$. Then $t\to\infty$ as $n\to\infty$. Also taking logarithms, we have

$$n\log(1/x)=\log t,\ \text{ or } n=\frac{\log t}{\log(1/x)}.$$

Therefore $nx^n = \dfrac{n}{(1/x)^n} = \dfrac{\log t}{t \log(1/x)}$.

Also $(\log t)/t \to 0$ as $t \to \infty$, by (3) above, and $\log(1/x)$ remains constant when n and t change. Therefore

$$\lim_{n\to\infty} nx^n = 0.$$

7·3. Definition of Convergence. An infinite series

$$u_1 + u_2 + u_3 + \dots + u_n + \dots$$

is said to be *convergent* if s_n, the sum of its first n terms, tends to a finite limit as n tends to infinity.

If s_n tends to plus or minus infinity as n tends to infinity, the series is said to be *divergent*.

If s_n does not tend to a finite limit, or to plus or minus infinity, the series is said to be *oscillatory*.

In the last case the value of s_n fluctuates, either within a finite range, or the numerical value of s_n tends to infinity while its sign is alternately positive and negative. In the former case the series is said to *oscillate finitely*; and in the latter case it is said to *oscillate infinitely*.

When a series is convergent and

$$\lim s_n = s,$$

s is called the sum of the series to infinity.

Series which diverge or oscillate are often said to be non-convergent.*

*The student should note that the words 'divergent' and 'oscillatory' are used differently by different writers. Some writers regard 'divergent' as equivalent to 'non-convergent'; while some call a series 'oscillatory' only if it oscillates *finitely*, series which oscillate infinitely being classed as 'divergent'. But the classification given in this book is the one generally used.

Ex. Show that the series $1+r+r^2+\ldots+r^{n-1}+\ldots (r>0)$ is convergent if $r<1$ and divergent if $r\geqslant 1$.

We have $s_n=(1-r^n)/(1-r)$.

If $r<1$, $r^n\to 0$ as $n\to\infty$, and therefore $s_n\to 1/(1-r)$. Hence the series is convergent for $r<1$.

If $r>1$, we can write $s_n=(r^n-1)/(r-1)$. As $n\to\infty$, $r^n\to\infty$, and therefore s_n also $\to\infty$.

Hence the series is divergent for $r>1$.

If $r=1$, the series becomes $1+1+1+\ldots$, and $s_n=n$. As $n\to\infty$, $s_n\to\infty$ and the series is divergent.

7·31. Some general theorems. We shall find the following general theorems on infinite series useful.

(1) If the series $u_1+u_2+\ldots$ is convergent and has the sum s, then the series $a+u_1+u_2+\ldots$ is convergent and has the sum $a+s$. Similarly the series $a+b+\ldots+k+u_1+u_2+\ldots$ is convergent and has the sum $a+b+\ldots+k+s$.

(2) If the series $u_1+u_2+\ldots$ is convergent and has the sum s, the series $u_{m+1}+u_{m+2}+\ldots$ is also convergent and has the sum $s-u_1-u_2-\ldots-u_m$.

(3) If the series $u_1+u_2+\ldots$ diverges or oscillates, then the other series considered in (1) and (2) do the same.

(4) If the series $u_1+u_2+\ldots$ is convergent and has the sum s, then the series $ku_1+ku_2+\ldots$ is also convergent and has the sum ks.

(5) If the first series considered in (4) diverges or oscillates, so does the second, unless $k=0$.

The above theorems are almost obvious and may be proved at once from the definitions. We can also enunciate them in the following way.

The nature of an infinite series remains unaltered by the addition or removal of a finite number of terms, or by the multiplication of each term by a fixed number $(\neq 0)$.

(6) *The nature of an infinite series of* POSITIVE *terms in unaltered by grouping the terms in brackets in any way to form new single terms.*

The theorem may not be true for a series whose terms are not positive. For example, the oscillatory series

$$1-1+1-1+\ldots$$

can be grouped as

$$(1-1)+(1-1)+\ldots, \text{ i.e., } 0+0+\ldots,$$

which is a convergent series.

Examples

Find the limit, as n tends to infinity, of the following :

1. $(n^2+1)/(n+1)(n+2)$.
2. $\sqrt[3]{(3n^4+n^2+3)}/2n(n+2)^{1/3}$.
3. $\sqrt[3]{(2n^2+3)}/\sqrt{(n^3+4)}$.
4. $(n^2+4n+1)/(5n+3)$.
5. n^3e^{-n}.
6. $(\log n)^2/\sqrt{n}$.
7. Show that for all values of x

$$\lim_{n\to\infty}\frac{x^n}{n!}=0. \qquad \text{[Annamalai, 1950]}$$

8. Evaluate

(i) $\lim_{n\to\infty}[\sqrt{\{n(n+1)\}}-n]$.

(ii) $\lim_{n\to\infty} n\{\log(n+1)-\log n\}$. [*Mysore*, '48-49]

By considering the sum of the first n terms, determine whether the following series are convergent or divergent:

9. $1+\frac{3}{4}+\frac{9}{16}+\frac{27}{64}+\ldots$.
10. $1+2+3+4+\ldots$.
11. $1+2x+3x^2+4x^3+\ldots$, if $|x|<1$.
12. $\dfrac{1}{(m+1)(m+2)}+\dfrac{1}{(m+2)(m+3)}+\dfrac{1}{(m+3)(m+4)}+\ldots$.

Series of Positive Terms

7·4. Test for Convergence. We cannot find the sum of the first n terms of every series. Therefore the definition of convergence cannot be

applied directly in every case, and we have to devise other methods for testing the convergence of a given series.

First of all, we shall take up tests for a series all of whose terms are positive. Most of these tests depend upon the following important theorem.

A series of positive terms is convergent if s_n, the sum to n terms, is less than a fixed number for all values of n.

Let the series be $u_1+u_2+u_3+\ldots$, where $u_1, u_2, \ldots$ are all positive. Then s_n, i.e., $u_1+u_2+\ldots+u_n$, will go on increasing as n increases, and may tend to a finite limit or to plus infinity. It cannot oscillate.

Therefore, if s_n remains less than a fixed number for all values of n, it cannot tend to infinity, and so must tend to a finite limit. Hence the given series is convergent.

Ex. Show that the series $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots$ is convergent.

For this series
$$s_n=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2}$$
$$<1+\frac{1}{1.2}+\frac{1}{2.3}+\ldots+\frac{1}{(n-1)n},$$
$$<1+\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\ldots+\left(\frac{1}{n-1}-\frac{1}{n}\right),$$
$$<2-1/n,$$
showing that s_n is less than 2 for all values of n.

Therefore the series is convergent.

7·41. All terms greater than a fixed number. *If in an infinite series of positive terms every term is greater than a fixed positive number, the series is divergent.*

Let every term be greater than the fixed positive number a. Then the sum of the first n terms is greater than na, and by taking n sufficiently large this can be made to exceed any finite number. Therefore the series is divergent.

The student should note that the proposition is true even if a is very small, but a must be (i) positive and (ii) fixed, that is, independent of n.

COROLLARY. *A series of positive terms is divergent if*

$$\lim u_n > 0.$$

Let $\lim u_n$ be l, where $l > 0$, and take a positive number a less than l. Then, leaving aside a finite number of terms, every term will be greater than a (§ 7·21, ii). Therefore the series is divergent.

It follows that *for every convergent series*

$$\lim u_n = 0.$$

The converse of this is not true. If $\lim u_n = 0$, the series may or may not be convergent.

For example, consider the series

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \dots + \frac{1}{\sqrt{n}} + \dots.$$

Since the terms go on decreasing,

$$s_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} + \dots + \frac{1}{\sqrt{n}},$$

i.e., $s_n > \sqrt{n}$, which tends to infinity as n tends to infinity. Therefore the series is divergent, even though $\lim u_n = 0$.

Thus, *in order that Σu_n be convergent it is necessary, but not sufficient, that*

$$\lim u_n = 0.$$

7·42. Ratio Test. *An infinite series of positive terms is convergent if from and after some term the ratio of each term to the preceding term is less than a fixed number which is less than unity.*

The series is divergent if the above ratio is greater than or equal to unity.

(i) Let the series beginning from the specified term be denoted by

$$u_1+u_2+u_3+u_4+\ldots$$

and let $$\frac{u_2}{u_1}<r,\ \frac{u_3}{u_2}<r,\ \frac{u_4}{u_3}<r,\ \ldots,$$

where $r<1$.

Then $$u_1+u_2+u_3+u_4+\ldots$$

$$=u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_1}+\frac{u_4}{u_1}+\ldots\right)$$

$$=u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\frac{u_4}{u_3}\cdot\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\ldots\right)$$

$$<u_1(1+r+r^2+r^3+\ldots),$$

that is, $$\Sigma u_n<u_1/(1-r), \text{ since } r<1.$$

Hence the given series is convergent.

(ii) Again if $\dfrac{u_2}{u_1}\geqslant 1,\ \dfrac{u_3}{u_2}\geqslant 1,\ \dfrac{u_4}{u_3}\geqslant 1,\ \ldots,$

then $$u_2\geqslant u_1,\ u_3\geqslant u_1,\ u_4\geqslant u_1,\ \ldots.$$

If s_n is the sum of the first n terms, it follows that

$$s_n\geqslant nu_1.$$

By taking n sufficiently large, we see that nu_1, and therefore also s_n, can be made greater than any finite number. Hence the given series is divergent in this case.

The above test is known as the ratio test and also as D'Alembert's test.*

Corollary. *The series Σu_n is convergent if*

$$\lim \frac{u_{n+1}}{u_n} < 1,$$

and divergent if lim $u_{n+1}/u_n > 1$.

Let lim u_{n+1}/u_n be l, and consider first the case when $l<1$. Choose a positive number b lying between l and 1. Then b is greater than l. Therefore, for all terms, except a finite number of them,

$$u_{n+1}/u_n < b$$

(by § 7·21, ii). But b is less than 1. Thus u_{n+1}/u_n is less than a fixed number which is less than unity. Hence the series is convergent.

The proof when $l>1$ is similar.

When lim $u_{n+1}/u_n=1$, the test fails; and we cannot say whether the series is convergent or divergent.

Note. The student often wonders why it is necessary that u_{n+1}/u_n should be less than a fixed number which is less than unity. He thinks that this statement is equivalent to saying that u_{n+1}/u_n is always less than unity. But this is not so. If u_{n+1}/u_n is always less than unity, the series may or may not be convergent.

For example in the two series

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\dots+\frac{1}{\sqrt{n}}+\dots \quad \text{and} \quad 1+\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{n^2}+\dots,$$

u_{n+1}/u_n is equal to

$$\sqrt{\left(\frac{n}{n+1}\right)} \quad \text{and} \quad \left(\frac{n}{n+1}\right)^2$$

*Called after the French mathematician Jean-le-Rond D' Alembert (1717-1783). He is more famous for his researches in dynamics.

respectively, both of which are less than 1. But for both of these there is no fixed number l such that

$$u_{n+1}/u_n < l < 1$$

for every value of n. Now the first of these two series is divergent (§ 7·41), while the second is convergent (§ 7·4), which shows that it is not sufficient for convergence that u_{n+1}/u_n be less than unity. It may be noted that for both the above series $\lim u_{n+1}/u_n = 1$, and so the results are not contrary to the corollary.

In applying the ratio test, it is more convenient to find $\lim u_{n+1}/u_n$, than to find a number less than 1 and exceeding each ratio. If this limit is unity, then further tests should be applied.

Ex. 1. Test for convergence the series

$$\frac{x}{1.2}+\frac{x^2}{3.4}+\frac{x^3}{5.6}+\frac{x^4}{7.8}+\ldots, \text{ where } x>0.$$

Here $u_n = x^n/(2n-1)2n$ and $u_{n+1} = x^{n+1}/(2n+1)(2n+2)$.

$$\text{Therefore } \lim\frac{u_{n+1}}{u_n} = \lim\frac{(2n-1)2n}{(2n+1)(2n+2)}x$$

$$= \lim\frac{1-1/2n}{(1+1/2n)(1+2/2n)}x = x.$$

Hence if $x<1$, the series is convergent; and if $x>1$, the series is divergent. If $x=1$, $\lim u_{n+1}/u_n = 1$, and further tests are necessary. [It will be shown later that the series is convergent in this case.]

Ex. 2. Find whether the series

$$2+\tfrac{3}{2}x+\tfrac{4}{3}x^2+\tfrac{5}{4}x^3+\ldots$$

is convergent or divergent for positive values of x.

We have $u_n = (n+1)x^{n-1}/n$, and $u_{n+1} = (n+2)x^n/(n+1)$.

$$\text{Therefore} \quad \lim\frac{u_{n+1}}{u_n} = \lim\frac{n(n+2)}{(n+1)^2}x = x.$$

Hence if $x<1$, the series is convergent; and if $x>1$, the series is divergent. If $x=1$, the series becomes $2+\frac{3}{2}+\frac{4}{3}+\frac{5}{4}+\ldots$, which is divergent since every term is greater than 1.

7·43. Root Test. *A series Σu_n of positive terms is convergent if for every value of n, $\sqrt[n]{u_n}$ is less than a fixed number which is less than unity.*

The series is divergent if $\sqrt[n]{u_n} \geqslant 1$ for every value of n.

(i) Let $\sqrt[n]{u_n} < r$, where $r < 1$.

Then $u_n < r^n$, for all values of n. Therefore

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots < r + r^2 + r^3 + \ldots + r^n + \ldots,$$

that is $\Sigma u_n < r/(1-r)$, which is a fixed number.

Hence Σu_n is convergent.

(ii) If $\sqrt[n]{u_n} \geqslant 1$, then $u_n \geqslant 1$ for all values of n. Hence the series is divergent.

NOTE. The theorem is true even if the condition holds from and after a fixed term.

COROLLARY. *The series Σu_n is convergent if*

$$\lim \sqrt[n]{u_n} < 1,$$

and divergent if $\lim \sqrt[n]{u_n} > 1$.

If $\lim \sqrt[n]{u_n} = 1$, the test fails.

The proof is similar to that of the corollary in the last article.

Ex. Test for convergence the series

$$1 + \frac{x}{2} + \frac{x^2}{3^2} + \frac{x^3}{4^3} + \ldots, \text{ where } x > 0.$$

Consider the series obtained on omitting the first term; for it

$$u_n = x^n/(n+1)^n,$$

so that

$$\lim \sqrt[n]{u_n} = \lim x/(n+1) = 0.$$

Hence the series is convergent for all values of x.

EXAMPLES

Determine whether the following series are convergent or divergent.

1. $\dfrac{1}{\sqrt{2}} + \dfrac{2}{\sqrt{5}} + \dfrac{4}{\sqrt{17}} + \dfrac{8}{\sqrt{65}} + \ldots + \dfrac{2^n}{\sqrt{(4^n+1)}} + \ldots.$

2. $\frac{1}{1+2^{-1}}+\frac{2}{1+2^{-2}}+\frac{3}{1+2^{-3}}+\dots$. [*Andhra*, 1948]

3. $\frac{1}{1+2}+\frac{2}{1+2^2}+\frac{3}{1+2^3}+\dots$.

4. $1+\frac{3}{2!}+\frac{5}{3!}+\frac{7}{4!}+\dots$.

5. $1+\frac{2^p}{2!}+\frac{3^p}{3!}+\frac{4^p}{4!}+\dots$. [*Gauhati*, 1955]

Test for convergence the following series for positive value of x.

6. $1+3x+5x^2+7x^3+\dots$.

7. $1^2+2^2x+3^2x^2+4^2x^3+\dots$.

8. $x+\frac{3}{5}x^2+\frac{8}{10}x^3+\frac{15}{17}x^4+\dots+\frac{n^2-1}{n^2+1}x^n+\dots$. [*Raj.*, '60]

9. $1+\frac{2}{5}x+\frac{6}{9}x^2+\frac{14}{17}x^3+\dots+\frac{2^n-2}{2^n+1}x^{n-1}+\dots$. [*Baroda*,'60]

10. $\frac{1}{2}+\frac{2}{3}x+(\frac{3}{4})^2x^2+(\frac{4}{5})^3x^3+\dots$.

7·44. Comparison Test. *If Σu_n and Σv_n are two series of positive terms and k is a fixed positive number, then (i) Σu_n will be convergent if Σv_n is convergent and u_n/v_n is always less than k; and (ii) Σu_n will be divergent if Σv_n is divergent and u_n/v_n is always greater than k.*

Let s_n and t_n denote the sums of the first n terms of Σu_n and Σv_n respectively.

(i) Then since by hypothesis

$$\frac{u_1}{v_1}<k,\ \frac{u_2}{v_2}<k,\ \dots,\ \frac{u_n}{v_n}<k,\ \dots,$$

therefore $u_1+u_2+\dots+u_n<k(v_1+v_2+\dots+v_n)$,

i.e., $$s_n<kt_n.$$

Now Σv_n is convergent, so t_n tends to a finite limit t as n tends to infinity. Therefore $s_n < kt$ for all values of n. Hence Σu_n is convergent (§ 7·4).

(ii) If Σu_n is divergent and

$$\frac{u_1}{v_1} > k, \frac{u_2}{v_2} > k, \ldots, \frac{u_n}{v_n} > k, \ldots,$$

then
$$s_n > kt_n.$$

As $n \to \infty$, $t_n \to \infty$. Therefore s_n also tends to infinity and Σu_n is divergent.

COROLLARY 1. *If Σu_n and Σv_n are two series of positive terms, then (i) Σu_n will be convergent if Σv_n is convergent and $u_n < v_n$ for all values of n; and (ii) Σu_n will be divergent if Σv_n is divergent and $u_n > v_n$ for all values of n.*

This is obtained by putting $k = 1$ in the above.

COROLLARY 2. *If Σu_n and Σv_n are two series of positive terms and u_n/v_n always lies between two fixed positive (non-zero) numbers, then the series are both convergent or both divergent.*

If k_1 and k_2 are the two numbers, then, as before,

$$k_1(v_1 + v_2 + \ldots + v_n) < (u_1 + u_2 + \ldots + u_n) < k_2(v_1 + v_2 + \ldots + v_n).$$

Therefore, if t_n tends to a finite limit, so does s_n; and if t_n tends to infinity, s_n also tends to infinity. Hence the proposition.

The student should note that the condition that u_n/v_n always remains non-zero and finite is by itself not sufficient to ensure that the series are either both convergent or both divergent. Consider, for example, the series

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \text{ and } 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \ldots .$$

The ratio u_n/v_n is $1/n\sqrt{n}$, which is finite and non-zero for every value of n; similarly the ratio v_n/u_n, i.e. $n\sqrt{n}$, is finite for every value of n. But the first of the two series is convergent and the second divergent.

COROLLARY 3. *If Σu_n and Σv_n are two series of positive terms and lim u_n/v_n is non-zero and finite, the two series are both convergent or divergent.*

This follows from the above corollary on applying § 7·21 (ii).

The comparison test can be applied to determine the convergence or divergence of a given series, provided we have a suitable *auxiliary* series whose convergence or divergence has already been established. The series discussed in the next article will be found useful for this purpose.

7·45. The series $\Sigma 1/n^p$. *The infinite series*

$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots+\frac{1}{n^p}+\ldots$$

is **convergent** *if* $\mathbf{p>1}$, *and* **divergent** *if* $\mathbf{p\leqslant 1}$.

CASE I. Let $p>1$. As the terms are all positive, we can group them as we like. Now group them as follows:

$$\frac{1}{1^p}+\left(\frac{1}{2^p}+\frac{1}{3^p}\right)+\left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right)$$
$$+\left(\frac{1}{8^p}+\frac{1}{9^p}+\ldots+\frac{1}{15^p}\right)+\ldots. \qquad (1)$$

This series is less, term by term, than the series

$$\frac{1}{1^p}+\left(\frac{1}{2^p}+\frac{1}{2^p}\right)+\left(\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}\right)$$
$$+\left(\frac{1}{8^p}+\frac{1}{8^p}+\ldots+\frac{1}{8^p}\right)+\ldots. \qquad (2)$$

But (2) can be written as

$$1+\frac{2}{2^p}+\frac{4}{4^p}+\frac{8}{8^p}+\dots,$$

which is a geometric series with the common ratio $2/2^p=1/2^{p-1}$. Also $1/2^{p-1}<1$. So the series (2) is convergent. Therefore the series (1) is also convergent.

CASE II. Let $p=1$. Then the series becomes $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots$. The terms in this series can be grouped as

$$1+\tfrac{1}{2}+(\tfrac{1}{3}+\tfrac{1}{4})+(\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8})+(\tfrac{1}{9}+\dots+\tfrac{1}{16})+\dots (3)$$

This is term by term greater than

$$1+\tfrac{1}{2}+(\tfrac{1}{4}+\tfrac{1}{4})+(\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8})+(\tfrac{1}{16}+\dots+\tfrac{1}{16})+\dots$$

i.e.,
$$1+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\dots,$$

which is divergent. Therefore (3) is also divergent.

CASE III. Let $p<1$ (negative values of p being included). Then the series is term by term greater than the series (3), and is, therefore, divergent.

7·46. Examples on Comparison Test. In applying the comparison test to a given series Σu_n, it is more convenient to find $\lim u_n/v_n$ than to find a fixed number exceeding u_n/v_n for every value of n. The auxiliary series Σv_n must be chosen in such a way that $\lim u_n/v_n$, is non-zero and finite. This can generally be done by taking v_n equal to the term of the highest degree in n (or the lowest degree in $1/n$) occurring in u_n.

Ex. 1. Test for convergence $\sum_{n=1}^{\infty}[\sqrt{(n^2+1)}-n]$.

[*Allahabad*, 1954]

Here $u_n = \sqrt{(n^2+1)} - n = n(1+1/n^2)^{1/2} - n$

$= n\{1+\frac{1}{2}(1/n^2) - \frac{1}{8}(1/n^4) + \ldots\} - n$

$= \frac{1}{2}(1/n) - \frac{1}{8}(1/n^3) + \ldots$.

Take $v_n = 1/n$; then

$$\lim u_n/v_n = \lim \{\tfrac{1}{2} - \tfrac{1}{8}(1/n^2) + \ldots\} = \tfrac{1}{2},$$

which is finite and non-zero. Since Σv_n is divergent, therefore Σu_n is also divergent.

Ex. 2. Test the convergence of the series

$$\frac{1}{2\sqrt{1}} + \frac{x^2}{3\sqrt{2}} + \frac{x^4}{4\sqrt{3}} + \frac{x^6}{5\sqrt{4}} + \ldots \text{to } \infty. \qquad [\textit{Agra}, 1955]$$

Here $u_n = x^{2n-2}/(n+1)\sqrt{n}$ and $u_{n+1} = x^{2n}/(n+2)\sqrt{(n+1)}$.

Therefore $\lim \dfrac{u_{n+1}}{u_n} = \lim \dfrac{(n+1)\sqrt{n}}{(n+2)\sqrt{(n+1)}} x^2 = x^2$.

Hence the series is convergent if $x^2 < 1$, and divergent if $x^2 > 1$.

If $x^2 = 1$, $u_n = 1/(n+1)\sqrt{n}$; and taking $v_n = 1/n^{3/2}$, we find that $\lim u_n/v_n = 1$. Since Σv_n is convergent, the given series also is convergent for $x^2 = 1$.

EXAMPLES

Determine whether the following series are convergent or divergent.

1. $1 + \dfrac{1}{3} + \dfrac{1}{5} + \dfrac{1}{7} + \ldots$.

2. $\sqrt{\dfrac{1}{2^3}} + \sqrt{\dfrac{2}{3^3}} + \sqrt{\dfrac{3}{4^3}} + \ldots$. [*Aligarh*, 1960]

3. $\dfrac{1}{1.2.3} + \dfrac{3}{2.3.4} + \dfrac{5}{3.4.5} + \ldots$. [*Sagar*, 1949]

4. $1 + \dfrac{1}{2^2} + \dfrac{2^2}{3^3} + \dfrac{3^3}{4^4} + \dfrac{4^4}{5^5} + \ldots$. [*Kashmir*, 1954]

5. $\dfrac{2}{1^p} + \dfrac{3}{2^p} + \dfrac{4}{3^p} + \dfrac{5}{4^p} + \ldots$. [*Agra*, 1950]

Test for convergence the series whose general term is:

6. $\dfrac{\sqrt{n}}{n^2+1}$. [*Ban.*, '53]

7. $\dfrac{n}{(a+nb)^2}$.

8. $\sqrt[3]{(n^3+1)}-n$. [*Baroda*, '60]

9. $\sqrt{(n^4+1)}-\sqrt{(n^4-1)}$. [*Aligarh*, 1956]

Test the convergence of the following series for $x>0$.

10. $1+\frac{1}{2}x+\frac{1}{5}x^2+\frac{1}{10}x^3+\ldots+x^n/(n^2+1)+\ldots$. [*Delhi*, '51]

11. $\dfrac{x}{1.2}+\dfrac{x^2}{2.3}+\dfrac{x^3}{3.4}+\dfrac{x^4}{4.5}+\ldots$. [*Patna*, 1957]

12. $\dfrac{x}{x+1}+\dfrac{x^2}{x+2}+\dfrac{x^3}{x+3}+\ldots$.

7·47. Comparison of Ratios. *If Σu_n and Σv_n be two series of positive terms, then Σu_n is convergent if (i) Σv_n is convergent, and (ii) from and after some particular term*

$$\frac{u_{n+1}}{u_n}<\frac{v_{n+1}}{v_n}.$$

Σu_n is divergent if (i) Σv_n is divergent, and (ii) from and after some particular term $u_{n+1}/u_n>v_{n+1}/v_n$.

Since the removal of a finite number of terms does not alter the nature of the series, we shall suppose that the inequality is satisfied from the very first term.

I. Let $$\frac{u_2}{u_1}<\frac{v_2}{v_1},\ \frac{u_3}{u_2}<\frac{v_3}{v_2},\ \ldots.$$

Then $$u_1+u_2+u_3+\ldots+u_n=u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\ldots\right)$$

$$<u_1\left(1+\frac{v_2}{v_1}+\frac{v_3}{v_2}\cdot\frac{v_2}{v_1}+\ldots\right),$$

i.e., $$<\frac{u_1}{v_1}(v_1+v_2+v_3+\ldots+v_n).$$

Therefore, if Σv_n is convergent, Σu_n is also convergent.

II. Let $\dfrac{u_2}{u_1} > \dfrac{v_2}{v_1}, \ \dfrac{u_3}{u_2} > \dfrac{v_3}{v_2}, \ \ldots.$

Then $u_1 + u_2 + \ldots + u_n = u_1\left(1 + \dfrac{u_2}{u_1} + \dfrac{u_3}{u_2}\cdot\dfrac{u_2}{u_1} + \ldots\right)$

$$> u_1\left(1 + \frac{v_2}{v_1} + \frac{v_3}{v_2}\cdot\frac{v_2}{v_1} + \ldots\right),$$

i.e., $$> \frac{u_1}{v_1}(v_1 + v_2 + v_3 + \ldots + v_n).$$

Therefore, if Σv_n is divergent, Σu_n is also divergent.

COROLLARY. We can deduce from the above that Σu_n *is convergent if* (*i*) Σv_n *is convergent, and* (*ii*)

$$\lim \frac{u_{n+1}}{u_n} < \lim \frac{v_{n+1}}{v_n};$$

and Σu_n *is divergent if* (*i*) Σv_n *is divergent, and* (*ii*) $\lim u_{n+1}/u_n > \lim v_{n+1}/v_n$.

NOTE. The test for convergence given in the article above can also be stated as follows :

Σu_n *is convergent if* (*i*) Σv_n *is convergent, and* (*ii*) *from and after some particular term*

$$\frac{u_n}{u_{n+1}} > \frac{v_n}{v_{n+1}}.$$

This form of inequality is often more convenient to apply, as in the articles 7·48 and 7·49 below. The test for divergence can be stated similarly.

By taking $\Sigma 1/n^p$ for Σv_n and applying the above test, we can derive tests which are very convenient for determining convergence when $\lim u_{n+1}/u_n$ is unity.

7·48. Higher Ratio Test. *A series Σu_n of positive terms is convergent or divergent according as*

$$\lim \mathbf{n}\left(\frac{\mathbf{u}_n}{\mathbf{u}_{n+1}} - \mathbf{1}\right) > \mathbf{1}, \text{ or } < \mathbf{1}.$$

Let $$\lim n\left(\frac{u_n}{u_{n+1}} - 1\right) = l,$$

where $l>1$. Choose a number p lying between l and 1, and compare the given series Σu_n with the auxiliary series $\Sigma 1/n^p$, which is convergent since $p>1$.

Therefore, Σu_n will be convergent if from and after some particular term

$$\frac{u_n}{u_{n+1}} > \frac{(n+1)^p}{n^p} \text{ or } \left(1+\frac{1}{n}\right)^p,$$

that is, if $$\frac{u_n}{u_{n+1}} > 1 + \frac{p}{n} + \frac{p(p-1)}{2n^2} + \dots,$$

or $$n\left(\frac{u_n}{u_{n+1}} - 1\right) > p + \frac{p(p-1)}{2n} + \dots. \tag{1}$$

By taking n sufficiently large the left-hand side can be made to approach l as nearly as we please, and the right-hand side can be made to approach p as nearly as we please. Also, l is greater than p. Therefore for all sufficiently large values of n, the inequality (1) is satisfied. Hence Σu_n is convergent.

The other case, when $l<1$, can be proved similarly. This test is known as Raabe's test*

NOTE 1. The above test should be applied only in those cases in which $\lim u_{n+1}/u_n = 1$, and so the simple ratio test (§ 7·42) fails. When the present test also fails, further tests are necessary. It will be shown in § 7·53, however, that in such cases the series is divergent, provided that u_n/u_{n+1} does not involve n as a logarithm or an exponent.

*Joseph Ludwig Raabe (1801—1859) of Zurich was the first to evolve these more delicate tests for convergence.

NOTE 2. The student should note that for the simple ratio test the limit of u_{n+1}/u_n was determined, while the present and all the subsequent tests (§§7·49, 7·52, 7·53) employ the ratio $\mathbf{u_n/u_{n+1}}$.

Ex. Test for convergence the series

$$1+\frac{3}{7}x+\frac{3.6}{7.10}x^2+\frac{3.6.9}{7.10.13}x^3+\frac{3.6.9.12}{7.10.13.16}x^4+\ldots.$$

Here $$u_n=\frac{3.6.9\ldots(3n-3)}{7.10.13\ldots(3n+1)}x^{n-1}$$

and $$u_{n+1}=\frac{3.6.9\ldots(3n-3)3n}{7.10.13\ldots(3n+1)(3n+4)}x^n.$$

Therefore $$\frac{u_{n+1}}{u_n}=\frac{3n}{3n+4}x, \text{ and } \lim\frac{u_{n+1}}{u_n}=x.$$

Hence the series is convergent if $x<1$ and divergent if $x>1$ (considering only positive values of x). If $x=1$,

$$\lim n\left(\frac{u_n}{u_{n+1}}-1\right)=\lim n\left(\frac{3n+4}{3n}-1\right)=\frac{4}{3},$$

which is greater than 1. Hence the series is convergent.

7·49. Logarithmic Ratio Test. *A series Σu_n of positive terms is convergent or divergent according as*

$$\mathbf{\lim\left\{n\log\frac{u_n}{u_{n+1}}\right\}>1, \text{ or } <1.}$$

Let $$\lim n\log(u_n/u_{n+1})=l,$$

where $l>1$, and choose a number p lying between l and 1. Then, comparing the given series Σu_n with the auxiliary series $\Sigma 1/n^p$, which is convergent since $p>1$, we see that Σu_n is also convergent if from and after some particular term

$$\frac{u_n}{u_{n+1}}>\frac{(1+n)^p}{n^p} \text{ or } \left(1+\frac{1}{n}\right)^p;$$

or, taking the logarithms of both the sides,

if $$\log \frac{u_n}{u_{n+1}} > p \log\left(1+\frac{1}{n}\right),$$

that is, if $$\log \frac{u_n}{u_{n+1}} > p\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots\right),$$

or $$n \log \frac{u_n}{u_{n+1}} > p + \frac{p}{n}\left(-\frac{1}{2} + \frac{1}{3n} - \ldots\right).$$

By taking n sufficiently large, the left-hand side can be made to approach l as nearly as we please, and the right-hand side can be made to approach p as nearly as we please. Also, l is greater than p. Therefore the above inequality is satisfied for all sufficiently large values of n. Hence Σu_n is convergent.

The other case, when $l<1$, can be proved similarly.

This test is an alternative to Raabe's test and should be applied when $\lim u_n/u_{n+1}=1$, and taking the logarithm of u_n/u_{n+1} makes the evaluation of the limit easier.

Ex. Test for convergence and divergence the series

$$1+\frac{2x}{2!}+\frac{3^2x^2}{3!}+\frac{4^3x^3}{4!}+\frac{5^4x^4}{5!}+\ldots. \qquad \text{[\textit{Banaras}, 1960]}$$

Here $u_n = \dfrac{n^{n-1}x^{n-1}}{n!}$ and $u_{n+1} = \dfrac{(n+1)^n x^n}{(n+1)!}$.

Therefore $$\frac{u_{n+1}}{u_n} = \frac{(n+1)^{n-1}}{n^{n-1}} \cdot \frac{x}{n+1} = \left(1+\frac{1}{n}\right)^n \cdot \frac{n}{n+1} x,$$

and $$\lim (u_{n+1}/u_n) = ex.$$

Hence the series is convergent if $ex<1$, i.e., if $x<1/e$; and is divergent if $x>1/e$. If $x=1/e$, we have

$$\lim n \log \frac{u_n}{u_{n+1}} = \lim n \log \left\{\frac{e}{(1+1/n)^{n-1}}\right\}$$
$$= \lim n \{\log e - (n-1) \log (1+1/n)\}$$

$$=\lim n\left\{1-\frac{n-1}{n}+\frac{n-1}{2n^2}-\frac{n-1}{3n^2}+\dots\right\}$$

$$=\lim \{n-(n-1)+(\tfrac{1}{2}-1/2n)-(n-1)/3n^2+\dots\}$$

$$=1+\tfrac{1}{2}=\tfrac{3}{2},$$

which is greater than 1. Hence the series is convergent for $x=1/e$.

EXAMPLES

Find whether the following series are convergent or divergent.

1. $1+a+\dfrac{a(a+1)}{1.2}+\dfrac{a(a+1)(a+2)}{1.2.3}+\dots$ [*Agra*, 1950]

2. $\sum \dfrac{1^2.5^2.9^2\dots(4n-3)^2}{4^2.8^2.12^2\dots(4n)^2}$. [*Aligarh*, 1950]

3. $1+\dfrac{1}{2}\cdot\dfrac{x^2}{4}+\dfrac{1.3.5}{2.4.6}\cdot\dfrac{x^4}{8}+\dfrac{1.3.5.7.9}{2.4.6.8.10}\cdot\dfrac{x^6}{12}+\dots$

4. $x^2+\dfrac{2^2}{3.4}x^4+\dfrac{2^2.4^2}{3.4.5.6}x^6+\dfrac{2^2.4^2.6^2}{3.4.5.6.7.8}x^8+\dots$ [*Rajasthan*, 1959]

Test for positive values of x the convergence of the following series :

5. $1+\dfrac{1}{2}x+\dfrac{1.3}{2.4}x^2+\dfrac{1.3.5}{2.4.6}x^3+\dots$ [*Baroda*, 1959]

6. $x+\dfrac{2^2x^2}{2!}+\dfrac{3^3x^3}{3!}+\dfrac{4^4x^4}{4!}+\dfrac{5^5x^5}{5!}+\dots$ [*Rajasthan*, 1960]

7·5. Cauchy's Condensation Test.* *If the function $f(n)$ is positive for all positive integral values of n, and continually decreases as n increases, and if a is*

*Named after Augustin Louis Cauchy (1789-1857), a profound French mathematician. He was one of the leaders in infusing rigour in the treatment of infinite series.

a positive integer greater than unity, then the two series, $\Sigma f(n)$ and $\Sigma a^n f(a^n)$, are both convergent, or both divergent.

Write the first series $\Sigma f(n)$ as follows :

$$
\begin{aligned}
&\{f(1) \quad +f(2) \quad +\ldots+f(a)\} \\
&+\{f(a+1) \quad +f(a+2) \quad +\ldots+f(a^2)\} \\
&+\{f(a^2+1) \quad +f(a^2+2) \quad +\ldots+f(a^3)\}+\ldots. \quad (1)
\end{aligned}
$$

Then the terms in the rth group are

$$f(a^{r-1}+1)+f(a^{r-1}+2)+\ldots+f(a^r). \quad (2)$$

Each of these terms is greater than the last one, viz., $f(a^r)$, for the terms go on decreasing by hypothesis. Also the number of terms is a^r-a^{r-1}. Hence the sum of all the terms in (2) is greater than $(a^r-a^{r-1})\, f(a^r)$; that is,

$$f(a^{r-1}+1)+f(a^{r-1}+2)+\ldots+f(a^r)>(1-a^{-1})a^r f(a^r).$$

Giving to r the values 1, 2, 3, ..., n successively, and adding, we get

$$\sum_2^{a^n} f(r)>(1-a^{-1})\sum_1^n a^r f(a^r). \quad (3)$$

Again since the terms go on decreasing, each term in (2) is less than $f(a^{r-1})$, and their sum is less than $(a^r-a^{r-1})f(a^{r-1})$; that is,

$$f(a^{r-1}+1)+f(a^{r-1}+2)+\ldots+f(a^r)<(a-1)a^{r-1}f(a^{r-1}).$$

Putting r equal to 1, 2, 3, ..., n successively and adding, we get

$$\sum_2^{a^n} f(r)<(a-1)\sum_1^n a^{r-1}f(a^{r-1}). \quad (4)$$

The inequality (4) shows that if $\Sigma a^r f(a^r)$ is convergent, so is $\Sigma f(r)$; while (3) shows that if $\Sigma a^r f(a^r)$ is divergent so also is $\Sigma f(r)$.

Generally it is immaterial what value is assigned to a.

7·51. The series $\Sigma 1/n(\log n)^p$. *The infinite series*

$$1+\frac{1}{2(\log 2)^p}+\frac{1}{3(\log 3)^p}+\dots+\frac{1}{n(\log n)^p}+\dots$$

is convergent if $p>1$, and divergent if $p\leqslant 1$.

We shall apply Cauchy's condensation test to examine the convergence of the given series. In the present case

$$f(n)=1/n(\log n)^p,$$

and therefore

$$a^n f(a^n)=a^n/a^n(\log a^n)^p=1/(n\log a)^p$$
$$=(1/n^p)(\log a)^{-p}.$$

The constant factor $(\log a)^{-p}$ is common to every term. Also, $\Sigma 1/n^p$ is convergent if $p>1$ and divergent if $p\leqslant 1$. Therefore the given series also is convergent if $p>1$ and divergent if $p\leqslant 1$.

7·52. Another Ratio Test. The above series can be used to devise the following test which can be applied when $\lim n(u_n/u_{n+1}-1)=1$, and so Raabe's test fails.

A series Σu_n of positive terms is convergent or divergent according as

$$\lim\left\{\mathbf{n}\left(\frac{\mathbf{u_n}}{\mathbf{u_{n+1}}}-\mathbf{1}\right)-\mathbf{1}\right\}\log \mathbf{n}>\mathbf{1}, \text{ or } <\mathbf{1}.$$

Let
$$\lim\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n=l,$$

where $l>1$, and choose a number p lying between l and 1. Then the series $\Sigma 1/n(\log n)^p$ is convergent, since $p>1$. Comparing Σu_n with this series, we see that Σu_n will be convergent if

$$\frac{u_n}{u_{n+1}} > \frac{(n+1)\{\log(n+1)\}^p}{n(\log n)^p} \tag{1}$$

for all sufficiently large values of n.

Now, for large values of n

$$\log(n+1) = \log n + \log(1+1/n) = \log n + 1/n + \ldots,$$

and the right-hand side of (1) becomes

$$\left(1+\frac{1}{n}\right)\left(1+\frac{1}{n\log n}+\ldots\right)^p = \left(1+\frac{1}{n}\right)\left(1+\frac{p}{n\log n}+\ldots\right)$$

$$= 1+\frac{1}{n}+\frac{p}{n\log n}+ \text{ terms involving higher powers of } n \text{ and } \log n \text{ in the denominator.}$$

Hence the condition (1) can be written as

$$\frac{u_n}{u_{n+1}} > 1+\frac{1}{n}+\frac{p}{n\log n}+\ldots,$$

i.e.,

$$n\left(\frac{u_n}{u_{n+1}}-1\right) > 1+\frac{p}{\log n}+\ldots,$$

or $\left\{n\left(\dfrac{u_n}{u_{n+1}}-1\right)-1\right\}\log n > p +$ terms involving n and $\log n$ in the denominator.

Now the left-hand side tends to l and the right-hand side tends to p as $n \to \infty$, and $l > p$. Therefore the inequality is satisfied for large values of n. Hence Σu_n is convergent.

The other case, when $l < 1$, can be proved similarly.

Ex. Test the convergence of the series

$$1+\frac{2^2}{3^2}+\frac{2^2 \cdot 4^2}{3^2 \cdot 5^2}+\frac{2^2 \cdot 4^2 \cdot 6^2}{3^2 \cdot 5^2 \cdot 7^2}+\ldots. \quad [\textit{Banaras}, 1960]$$

Here

$$u_n = \frac{2^2 \cdot 4^2 \ldots (2n-2)^2}{3^2 \cdot 5^2 \ldots (2n-1)^2},$$

and

$$u_{n+1} = \frac{2^2 \cdot 4^2 \ldots (2n-2)^2 (2n)^2}{3^2 \cdot 5^2 \ldots (2n-1)^2 (2n+1)^2}.$$

Therefore

$$\frac{u_{n+1}}{u_n} = \frac{(2n)^2}{(2n+1)^2},$$

which tends to 1 as $n \to \infty$. Proceeding to the next test, we see that

$$n\left(\frac{u_n}{u_{n+1}}-1\right)=n\frac{(2n+1)^2-(2n)^2}{(2n)^2}=\frac{4n+1}{4n},$$

which again tends to 1 as $n \to \infty$.

Finally, applying the test of the preceding article, we find that

$$\lim\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n=\lim\left\{\frac{4n+1}{4n}-1\right\}\log n$$
$$=\lim (\log n)/4n=0,$$

by (3), §7·23.

Since this limit is less than 1, the series is divergent.

This example is a particular case of the following general rule.

7·53. A general rule. *A series Σu_n of positive terms is* DIVERGENT *if*

$$n\left(\frac{u_n}{u_{n+1}}-1\right)$$

is an algebraic function of n which tends to 1 *as* $n \to \infty$.

Let $$n\left(\frac{u_n}{u_{n+1}}-1\right)=\frac{n^k+a_1n^{k-1}+\dots}{n^k+b_1n^{k-1}+\dots},$$

the terms of the highest degree in the numerator and denominator being the same, because the limit of the expression is 1.

Then $$\lim\left\{n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right\}\log n$$
$$=\lim\frac{(a_1-b_1)n^{k-1}+\dots}{n^k+b_1n^{k-1}+\dots}\log n$$
$$=\lim\frac{(a_1-b_1)n^k+\dots}{n^k+\dots}\cdot\frac{\log n}{n}.$$

Now the first factor tends to $a_1 - b_1$ and the second factor tends to zero as $n \to \infty$. Therefore the limit is zero, which is less than 1. Hence, by the theorem of the preceding article, Σu_n is divergent*.

An independent proof can be given by comparing Σu_n directly with $\Sigma 1/n(\log n)^p$.

Examples

Find whether the following series are convergent or divergent :

1. $\dfrac{(\log 2)^2}{2^2} + \dfrac{(\log 3)^2}{3^2} + \dots + \dfrac{(\log n)^2}{n^2} + \dots$.

2. $\dfrac{1}{(\log 2)^p} + \dfrac{1}{(\log 3)^p} + \dots + \dfrac{1}{(\log n)^p} + \dots$. [*Lkw.*, 1958]

3. $\dfrac{a}{b} + \dfrac{a(a+1)}{b(b+1)} + \dfrac{a(a+1)(a+2)}{b(b+1)(b+2)} + \dots$. [*Rajasthan*, 1958]

4. $1 + \dfrac{a(1-a)}{1^2} + \dfrac{(1+a)a(1-a)(2-a)}{1^2 . 2^2}$
$$+ \frac{(2+a)(1+a)a(1-a)(2-a)(3-a)}{1^2 . 2^2 . 3^2} + \dots.$$

Test the convergence of the following series for positive values of x:

5. $\dfrac{1^2}{2^2} + \dfrac{1^2 . 3^2}{2^2 . 4^3} x + \dfrac{1^2 . 3^2 . 5^2}{2^2 . 4^2 . 6^2} x^2 + \dots$. [*Baroda*, 1960]

6. $1 + \dfrac{\alpha . \beta}{1 . \gamma} x + \dfrac{\alpha(\alpha+1)\beta(\beta+1)}{1 . 2 . \gamma(\gamma+1)} x^2$
$$+ \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1 . 2 . 3 . \gamma(\gamma+1)(\gamma+2)} x^3 + \dots.$$ [*I.A.S.*, '59]

*The theorem can be proved similarly if $n(u_n/u_{n+1} - 1)$ involves fractional indices or radical sign. By an algebraic function we mean that it involves only powers of n, not its logarithm or exponential.

Series with terms Positive or Negative

7·6. Test for Alternating Series. *The infinite series $u_1-u_2+u_3-\dots$, in which the terms are alternately positive and negative, is convergent if each term is numerically less than the preceding term and lim $u_n=0$.*

Suppose $\qquad u_1>u_2>u_3>u_4>\dots,$

and $\lim u_n=0$. Consider the sum of an even number of terms. Then

$$s_{2n}=(u_1-u_2)+(u_3-u_4)+\dots+(u_{2n-1}-u_{2n}), \quad (1)$$

which can also be written as

$$s_{2n}=u_1-(u_2-u_3)-(u_4-u_5)-\dots-u_{2n}. \quad (2)$$

Since the terms are decreasing, the expressions within brackets in (1) and (2) are all positive. Therefore we see from (1) that s_{2n} is positive and increases as n increases; and we see from (2) that it always remains less than u_1. Hence s_{2n} must tend to a finite limit (§7·4).

Now consider the sum of an odd number of terms. Since

$$s_{2n+1}=s_{2n}+u_{2n+1}, \quad (3)$$

and $\lim u_{2n+1}=0$, therefore s_{2n+1} tends to the same limit as s_{2n}.

Hence s_n tends to a finite limit whether n is even or odd. Therefore the series is convergent.

Note. The student should note that it is necessary for convergence that $\lim u_n=0$. If $\lim u_n$ is not zero, we see from (3) that s_{2n+1} tends to a limit different from s_{2n}, and therefore the series is oscillatory. An example of such a series is $2-\frac{3}{2}+\frac{4}{3}-\frac{5}{4}+\dots$.

If the terms are alternately positive and negative, and go on increasing numerically, it can be shown that the series is oscillatory in this case also.

Ex. Show that the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ is convergent.

Here the terms are alternately positive and negative, each term is numerically less than the preceding term, and $\lim u_n$, i.e. $\lim (1/n)$ is zero. Hence the series is convergent.

7·61. Changing the sign of terms. If we take a convergent series of positive terms and make the sign of some of the terms negative, the new series thus obtained is convergent.

For, if we denote by s_n and t_n respectively the sums of the first n terms of the two series, then, numerically, t cannot be greater than s_n. Therefore t_n cannot tend to infinity. Also, $\lim u_n$ is zero. Therefore t_n cannot oscillate. Hence t_n must tend to a finite limit, and the series is convergent.

7·7. Absolute Convergence. Consider the series $u_1+u_2+\ldots+u_n+\ldots$ in which any term may be either positive or negative. Let $|u_n|$ denote the absolute value of u_n, that is, $|u_n|=u_n$ if u_n is positive and $|u_n|=-u_n$ if u_n is negative. Then

$$|u_1|+|u_2|+\ldots+|u_n|+\ldots$$

is a series each term of which is positive and numerically equal to the corresponding term of Σu_n.

If the series Σu_n is convergent, it is not necessary that $\Sigma|u_n|$ be also convergent. For example the series

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\ldots$$

is convergent, but the corresponding series of positive terms

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots$$

is divergent. To distinguish between the two cases which can arise we give the following definitions:

A series Σu_n containing positive and negative terms, is said to be *absolutely convergent* if the series $\Sigma|u_n|$ is convergent.

Σu_n is said to be *conditionally convergent* if Σu_n is convergent while $\Sigma|u_n|$ is divergent.

Thus the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$ is conditionally convergent, while the series

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots$$

is absolutely convergent.

It will be found that in a conditionally convergent series, the positive terms and the negative terms taken separately form two divergent series. When their terms are suitably arranged the difference gives a convergent series, while their sum naturally gives a divergent series. In an absolutely convergent series, the positive and negative terms both form convergent series.

By § 7·61 it follows that if $\Sigma|u_n|$ is convergent, Σu_n is also convergent. Therefore the absolute convergence of a series implies its ordinary convergence.

7·71. Some Well-Known Series. We discuss below the convergence of some well-known series.

I. BINOMIAL SERIES :

$$1+nx+\frac{n(n-1)}{2!}x^2+\ldots+\frac{n(n-1)\ldots(n-r+1)}{r!}x^r+\ldots, \quad (1)$$

when n is not a positive integer.

We have $$\frac{u_{r+1}}{u_r}=\frac{n-r+1}{r}x=-\left(1-\frac{n+1}{r}\right)x. \quad (2)$$

If x is positive, (2) is negative for large values of r and so the terms in (1) are alternately positive and negative ultimately. If x is negative, (2) is positive and ultimately

the terms in (1) are all of the same sign. It is most convenient, therefore, to test the series for absolute convergence.

Now, by (2), $|u_{r+1}|/|u_r| \to |x|$ as $r \to \infty$.

Therefore the series is absolutely convergent if $|x|<1$, i.e., if $-1<x<1$.

If $|x|>1$, we see from (2) that $|u_{r+1}/u_r|>1$ for all sufficiently large values of r. So the terms go on increasing numerically as r increases. Therefore the series is divergent (if the terms are ultimately of the same sign) or oscillatory (if the terms are alternately positive and negative).

When $x=-1$, the terms are ultimately of the same sign and $\lim u_{r+1}/u_r=1$. Therefore, applying Raabe's test, we get

$$\lim r\left(\frac{u_r}{u_{r+1}}-1\right)=\lim \frac{r(n+1)}{r-n-1}=n+1.$$

Hence the series is convergent for $n>0$, and divergent for $n<0$.

When $x=1$, the terms are alternately positive and negative for large values of r, and we can apply the test of § 7·6. It can be shown (see Ex., § 7·9) that $\lim u_r=0$ only when $n>-1$. Therefore the series is convergent for $n>-1$ and oscillatory for $n \leqslant -1$.

II. Exponential Series : $1+x+\dfrac{x^2}{2!}+\ldots+\dfrac{x^n}{n!}+\ldots$.

For this series $$\frac{|u_{n+1}|}{|u_n|}=\frac{|x|}{n},$$
which tends to zero as n tends to infinity. Thus
$$\lim|u_{n+1}|/|u_n|$$
is less than unity, whatever be the value of x. Hence the exponential series is absolutely convergent for every value of x.

The same is true for the series $1+x\log a+(x\log a)^2/2!+\ldots$

III. Logarithmic Series :

$$x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots+(-1)^{n-1}\frac{x^n}{n}+\ldots.$$

Here $$\frac{|u_{n+1}|}{|u_n|}=\frac{n}{n+1}\,|x|,$$
which tends to $|x|$ as $n\to\infty$. Therefore the series is absolutely convergent if $|x|<1$.

If $|x|>1$, the terms increase numerically as n increases, and so the series is divergent if x is negative, and oscillatory if x is positive.

If $x=1$, the series becomes $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots$, which is conditionally convergent.

If $x=-1$, the series becomes $-1-\frac{1}{2}-\frac{1}{3}-\frac{1}{4}-\ldots$, which is divergent.

7·72. Testing a given series. When it is required to test a given series for convergence, it is convenient to apply the tests in the following order.

A glance will show whether the given series is (i) a series with terms alternately positive and negative; (ii) a series of positive terms, which is not a power series*; or (iii) a power series of positive terms.

For series (i) the test for alternating series (§ 7·6) should be applied.

For series (ii) an inspection should be made of the nth term, and if feasible $\lim u_n$ should be determined. If $\lim u_n$ is not zero, the series is divergent. If $\lim u_n$ is zero, u_n may be comparable with $1/n^p$. If so, comparison test should be applied. If u_n involves $\log n$, Cauchy's condensation test can be applied.

In case the above tests are inapplicable, the ratio test should be applied. If this fails on account of $\lim u_{n+1}/u_n$ being 1, Raabe's test (§ 7·48) should be applied. If this also fails for a similar reason, and u_n/u_{n+1} is an algebraic function of n, the series is divergent (§ 7·53).

The logarithmic test (§ 7·49) should be applied as an alternative to Raabe's test when n occurs as an exponent in u_n/u_{n+1}. The root test (§ 7·43) is an alternative to the ratio test and should only be applied in special cases in which the nth root can be easily evaluated.

For the power series (iii) the ratio test should be applied first. If the series is in powers of x, it will be found

*A series in ascending powers of x (or some other number) is called a power series. Thus $a_0+a_1x+a_2x^2+a_3x^3+\ldots$, where $a_0, a_1, \ldots$ do not involve x, is a power series.

that $\lim u_{n+1}/u_n = lx$, where l is some constant. For $x<1/l$ the series will be convergent and for $x>1/l$, divergent. For $x=1/l$, the series should be treated just as in case (ii) above.

If a power series is to be tested for such values of x which make the terms alternately positive and negative, the results of § 7·6 should be applied.

Examples

Find whether the following series are convergent, divergent, or oscillatory.

1. $1-\dfrac{1}{2^2}+\dfrac{1}{3^2}-\dfrac{1}{4^2}+\dots$.

2. $\dfrac{1}{x}-\dfrac{1}{x+a}+\dfrac{1}{x+2a}-\dfrac{1}{x+3a}+\dots$.

3. $\left(\dfrac{1}{2}-\dfrac{1}{\log 2}\right)-\left(\dfrac{1}{2}-\dfrac{1}{\log 3}\right)+\left(\dfrac{1}{2}-\dfrac{1}{\log 4}\right)$
$-\left(\dfrac{1}{2}-\dfrac{1}{\log 5}\right)+\dots$. [*Annamalai*, 1949]

4. $\Sigma(-1)^n x^n/n$. [*Aligarh*, 1960]

5. (i) $\sum\dfrac{(-1)^n n}{2n+3}$; [*Kashmir*, '54] (ii) $\sum\dfrac{(-1)^n x^n}{\sqrt{n}}$.

6. Are the following series absolutely convergent ?

(i) $1-\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{3}}-\dfrac{1}{\sqrt{4}}+\dots$;

(ii) $1-\dfrac{1}{2\sqrt{2}}+\dfrac{1}{3\sqrt{3}}-\dfrac{1}{4\sqrt{4}}+\dots$;

(iii) $1-\dfrac{1}{2}+\dfrac{1}{4}-\dfrac{1}{8}+\dfrac{1}{16}-\dots$;

(iv) $1-2x+3x^2-4x^3+\dots$.

***7·8. Addition and Subtraction of Series.** *If the two series*

$$u_1+u_2+\dots \quad \textit{and} \quad v_1+v_2+\dots$$

are both convergent and their sums are s and t respectively, then
(i) $(u_1+v_1)+(u_2+v_2)+\dots$ is convergent and has the sum $s+t$; and
(ii) $(u_1-v_1)+(u_2-v_2)+\dots$ is convergent and has the sum $s-t$.

For (i) $\lim\{(u_1+v_1)+(u_2+v_2)+\dots+(u_n+v_n)\}$

$$=\lim\{(u_1+u_2+\dots+u_n)+(v_1+v_2+\dots+v_n)\}$$
$$=s+t. \quad (\S 7{\cdot}23, \text{i})$$

Hence $\Sigma(u_n+v_n)$ is convergent and has the sum $s+t$.

The proof of (ii) is similar.

7·81. Rearrangement of terms. (i) *If the terms of a convergent series of positive terms are rearranged, the series remains convergent and its sum is unaltered.*

Let $u_1+u_2+u_3+\dots$ be a convergent series, and after rearrangement let it become $v_1+v_2+v_3+\dots$. Denote the sums of the first n terms of the two series by s_n and t_n respectively, and let $\lim s_n=s$.

Consider the sum t_n, i.e.,

$$v_1+v_2+v_3+\dots+v_n. \tag{1}$$

Every term in this occurs somewhere in Σu_n. So by taking a sufficiently large number of terms from Σu_n we can include among them every term occurring in (1); that is, by taking m sufficiently large we can make

$$t_n<s_m.$$

But $s_m<s$. Therefore

$$t_n<s \tag{2}$$

for every value of n. Hence Σv_n converges to a limit (§7·4), say, to t. Also, on account of (2), $t\leqslant s$.

Again, since Σu_n can be obtained by rearranging the terms of Σv_n, we can prove similarly that $s\leqslant t$. Hence $t=s$.

(ii) *If the terms of an absolutely convergent series are rearranged, the series remains convergent and its sum is unaltered.*

*This and the subsequent articles may be omitted at a first reading.

Let $u_1+u_2+u_3+\dots$ be an absolutely convergent series and after rearrangement let it become $v_1+v_2+v_3+\dots$. Let the sum of the series Σv_n be s, and that of $\Sigma|u_n|$ be s'.

Then by § 7·8, $\Sigma(u_n+|u_n|)$ is convergent and has the sum $s+s'$.

Also, since Σv_n is a rearrangement of Σu_n, therefore $\Sigma|v_n|$ is a rearrangement of $\Sigma|u_n|$, and $\Sigma(v_n+|v_n|)$ is a rearrangement of $\Sigma(u_n+|u_n|)$.

But $\Sigma|u_n|$ and $\Sigma(u_n+|u_n|)$ are both series of positive or zero terms. Therefore, by (i) $\Sigma|v_n|$ and $\Sigma(v_n+|v_n|)$ are convergent and have the sums s' and $s+s'$ respectively.

Now Σv_n can be regarded as the difference of $\Sigma(v_n+|v_n|)$ and $\Sigma|v_n|$. Therefore Σv_n is convergent and has the sum $(s+s')-s'$, i.e., s.

(iii) The sum of a conditionally convergent series may get altered by a rearrangement of terms. In fact it can be proved that by a suitable rearrangement, the series can be made to approach any given number as its sum, and can be made even divergent. The reason is that in a conditionally convergent series the positive and negative terms taken separately form two divegent series. So if we take enough terms of one sign, their sum can be made as large as we please. Therefore, by decreasing in s_n the proportion of positive terms, the sum of the series gets decreased, and vice versa. This is illustrated in the example below.

Ex. Show that the sum of the series

$$1-\tfrac{1}{2}-\tfrac{1}{4}+\tfrac{1}{3}-\tfrac{1}{6}-\tfrac{1}{8}+\tfrac{1}{5}-\tfrac{1}{10}-\tfrac{1}{12}+\dots \tag{1}$$

is half the sum of the series

$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}+\tfrac{1}{7}-\tfrac{1}{8}+\dots. \tag{2}$$

The second series has the sum $\log_e 2$, as can be seen by putting $x=1$ in the expansion of $\log_e(1+x)$.

Let s_n denote the sum of the first n terms in (1), then

$$s_{3n}=\{(1-\tfrac{1}{2})-\tfrac{1}{4}\}+\{(\tfrac{1}{3}-\tfrac{1}{6})-\tfrac{1}{8}\}+\dots+\left\{\left(\frac{1}{2n-1}-\frac{1}{4n-2}\right)-\frac{1}{4n}\right\}$$

$$=\{\tfrac{1}{2}-\tfrac{1}{4}\}+\{\tfrac{1}{6}-\tfrac{1}{8}\}+\dots+\left\{\frac{1}{4n-2}-\frac{1}{4n}\right\}$$

$$=\tfrac{1}{2}(1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\dots\text{ to }2n\text{ terms}).$$

Therefore $\qquad \lim s_{3n} = \frac{1}{2}\log_e 2.$

Also, since $\lim u_n = 0$, s_{3n+1} and s_{3n+1} tend to the same limit.

Thus the series (1) has half the sum of (2), even though (1) is just a rearrangement of (2).

7·82. Multiplication of series. Consider the two series $a_1 + a_2x + a_3x^2 + \ldots$ and $b_1 + b_2x + b_3x^2 + \ldots$. If we multiply them out, the product, arranged in ascending powers of x, is

$$a_1b_1 + (a_1b_2 + a_2b_1)x + (a_1b_3 + a_2b_2 + a_3b_1)x^2 + \ldots .$$

Putting $x = 1$, and writing u and v for a and b, we see that we can write the formal product of Σu_n and Σv_n as

$$u_1v_1 + (u_1v_2 + u_2v_1) + (u_1v_3 + u_2v_2 + u_3v_1) + \ldots . \qquad (1)$$

But this formal product has a meaning only if series (1) is convergent. To study under what conditions it is so, we consider the series

$$(u_1v_1) + (u_1v_2 + u_2v_2 + u_2v_1) \\ + (u_1v_3 + u_2v_3 + u_3v_3 + u_3v_2 + u_3v_1) + \ldots , \qquad (2)$$

in which the first term is a product of u_1 and v_1; the terms in the first two pairs of brackets are the product of $u_1 + u_2$ and $v_1 + v_2$; the terms in the first three pairs of brackets are the product of $u_1 + u_2 + u_3$ and $v_1 + v_2 + v_3$; and so on.

Also, the series (2) is merely a rearrangement of the series (1). This can be seen from the two diagrams below, in both of which all the terms obtained by multiplying the terms of Σu_n by the terms of Σv_n are shown, but in Fig. 1 terms are

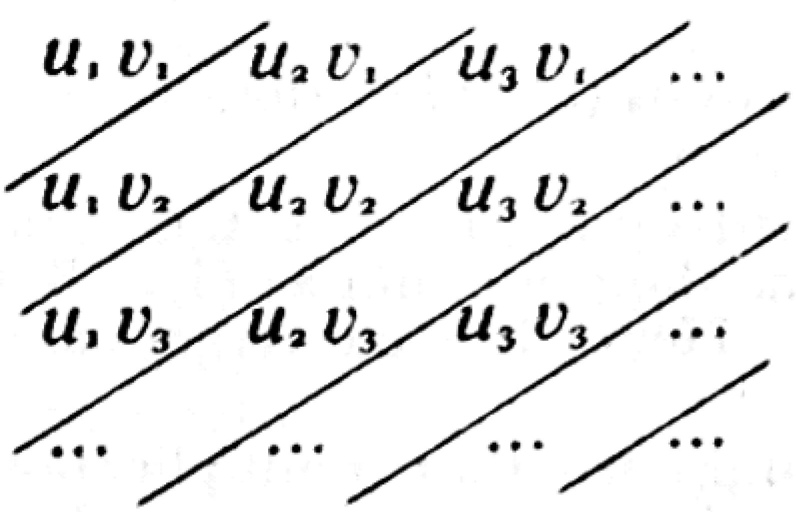


Fig. 1

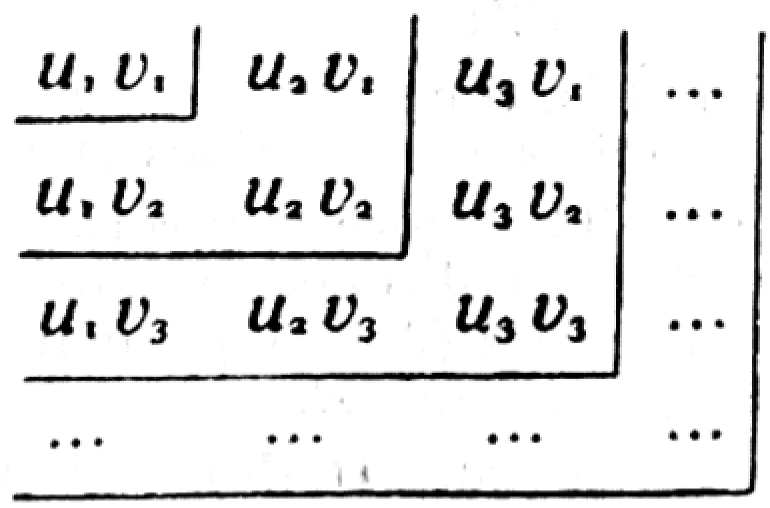


Fig. 2

taken diagonally, while in Fig. 2 they are taken parallel to the sides. The first arrangement gives the series (1), while the second gives the series (2).

We shall now prove the following proposition :

If $u_1+u_2+u_3+\ldots$ and $v_1+v_2+v_3+\ldots$ are convergent series of positive terms, or absolutely convergent series, and have the sums s and t, then the series

$$u_1v_1+(u_1v_2+u_2v_1)+(u_1v_3+v_2v_2+u_3v_1)+\cdots \quad (3)$$

is convergent and has the sum st.

Let s_n and t_n denote the sums of the first n terms of Σu_n and Σv_n respectively, and let s' and t' be the sums to infinity of the series $\Sigma|u_n|$ and $\Sigma|v_n|$.

Consider the series

$$u_1v_1+(u_1v_2+u_2v_2+u_2v_1)+(u_1v_3+u_2v_3+u_3v_3+u_3v_2+u_3v_1)+\ldots.(4)$$

The sum of the first n terms of this series, σ_n say is equal to s_nt_n. Now as n tends to infinity, $s_n\to s$ and $t_n\to t$; therefore σ_n tends to st.

Also, the series obtained from (4) by removing the brackets, namely

$$u_1v_1+u_1v_2+u_2v_2+u_2v_1+u_1v_3+u_2v_3+u_3v_3+u_3v_2+u_3v_1+\cdots \quad (5)$$

is absolutely convergent. For

$|u_1v_1|+|u_1v_2|+|u_2v_2|+|u_2v_1|+\ldots$ to n terms

$<(|u_1|+|u_2|+\ldots+|u_n|)(|v_1|+|v_2|+\ldots+|v_n|)$,

the right-hand side on multiplication containing more terms than the left-hand side,

$<s't'$;

that is, the sum of the fitrst n term of (5) is less than a fixed number for all values of n.

Therefore any series obtained by rearranging the terms of (5) is also convergent, and has the same sum as (4). But the series (3) is one such series. Hence it is convergent and has the sum st.

NOTE. It should not be supposed that multiplication of two convergent series always gives a convergent product series. When two conditionally convergent series are multi-

plied, the product series may or may not be convergent. It can be shown, however, that if one of the two series is absolutely convergent and the other conditionally convergent, the product series is convergent.

7·9. Infinite Products. Just as we have an infinite series $u_1+u_2+\ldots+u_n+\ldots$, we can have an infinite product

$$u_1u_2\ldots u_n\ldots. \tag{1}$$

In what follows we shall suppose that none of the factors is zero.

Let $$P_n=u_1u_2\ldots u_n;$$

then if P_n tends to a finite *non-zero* limit as n tends to infinity, we say that the infinite product (1) is convergent. The infinite product is said to be divergent if P_n tends to infinity. If P_n tends to zero, then also the infinite product is said to be divergent.

A full treatment of infinite products is outside the scope of this book. But many examples on infinite products can be solved by converting a product into a series by taking logarithms. Thus we can write

$$\log P_n=\log u_1+\log u_2+\ldots+\log u_n.$$

If $\Sigma \log u_n$ converges and has the sum s, then P_n converges to e^s; if $\Sigma\log u_n$ diverges to $+\infty$, P_n also diverges to $+\infty$, and if $\Sigma\log u_n$ diverges to $-\infty$, P_n diverges to zero.

Ex. Show that the limit of

$$\frac{n(n-1)(n-2)\ldots(n-r+1)}{r!},$$

as r tends to infinity, is zero, except when $n\leqslant-1$.

Rearrange the given product as

$$\left(\frac{n}{1}\right)\left(\frac{n-1}{2}\right)\left(\frac{n-2}{3}\right)\ldots\left(\frac{n+1}{r}-1\right). \tag{1}$$

For large values of r, $(n+1)/r<1$, so that the last factor is negative. Therefore the product becomes alternately positive and negative as r increases. Considering, however, the numerical value of the product and denoting it by P_r; we see that

$$\log P_r=\log\frac{|n|}{1}+\log\frac{|n-1|}{2}+\ldots+\log\left(1-\frac{n+1}{r}\right).$$

Denoting by u_r the rth term on the right-hand side, we have

$$u_r = \log\{1-(n+1)/r\} = -(n+1)/r-(n+1)^2/2r^2-\dots.$$

Comparing with the series $\Sigma 1/r$ we see that Σu_r diverges to $-\infty$ if $n+1$ is positive. Hence if $n>-1$, P_r tends to zero as r tends to infinity.

If $n<-1$, Σu_r diverges to $+\infty$ and P_r, the numerical value of (1), tends to infinity. If $n=-1$, then each factor in (1) is -1, and again P_r does not tend to zero.

EXAMPLES ON CHAPTER VII

Find whether the following series is convergent or divergent.

1. $\sqrt{\tfrac{1}{2}}+\sqrt{\tfrac{2}{3}}+\sqrt{\tfrac{3}{4}}+\sqrt{\tfrac{4}{5}}+\dots.$ [*Utkal*, 1950]

2. $1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\dots.$

3. $\dfrac{1}{1^p}+\dfrac{1}{3^p}+\dfrac{1}{5^p}+\dfrac{1}{7^p}+\dots.$ [*Allahabad*, 1950]

4. $1+\dfrac{2^2}{2!}+\dfrac{3^2}{3!}+\dfrac{4^2}{4!}+\dots.$

5. $\dfrac{1}{\sqrt{2}-1}+\dfrac{1}{\sqrt{3}-1}+\dfrac{1}{\sqrt{4}-1}+\dots.$ [*Rajputana*, 1950]

6. $\dfrac{1}{1+\sqrt{2}}+\dfrac{2}{1+2\sqrt{3}}+\dfrac{3}{1+3\sqrt{4}}+\dots$
$+\dfrac{n}{1+n\sqrt{(n+1)}}+\dots.$ [*Nagpur*, 1949]

7. $1^p+(\tfrac{1}{2})^p+\left(\dfrac{1.3}{2.4}\right)^p+\left(\dfrac{1.3.5}{2.4.6}\right)^p+\dots.$ [*Bombay*, 1954]

8. $\left(\dfrac{2^2}{1^2}-\dfrac{2}{1}\right)^{-1}+\left(\dfrac{3^3}{2^3}-\dfrac{3}{2}\right)^{-2}+\left(\dfrac{4^4}{3^4}-\dfrac{4}{3}\right)^{-3}+\dots.$
[*U.P.F.S.*, 1959]

Test for convergence the series whose nth term is:

9. $\dfrac{n^3+a}{2^n+a}$. [*Nagpur*, 1948]

10. $\dfrac{\sqrt{(n+1)}-\sqrt{n}}{n}$. [*Aligarh*, 1949]

11. $\sqrt{(n^3+1)}-\sqrt{n^3}$. [*Alig.*, '57]

12. $(1-1/n)^{n^2}$.

13. $\cos(1/n)$ [*Delhi*, 1958]

14. $\sin(1/n)$. [*Alld.*, 1957]

15. $(\log n)/n$.

16. $1/n \log n \log \log n$.

Test the convergence of the following series for positive value of x:

17. $\sum \dfrac{a^n}{a^n+x^n}$.

18. $\sum \dfrac{1}{x^n+x^{-n}}$. [*Bom.*, '52]

19. $\sum \dfrac{nx^n}{n^2+1}$. [*Andhra*, 1955]

20. $\sum \dfrac{3n+1}{4n+3}x^n$. [*Annamalai*, 1949]

21. $2x+\dfrac{3x^2}{8}+\dfrac{4x^3}{27}+\ldots+\dfrac{(n+1)x^n}{n^3}+\ldots$. [*Patna*, 1952]

22. $\dfrac{x}{1}+\dfrac{1}{2}\cdot\dfrac{x^3}{3}+\dfrac{1.3}{2.4}\cdot\dfrac{x^5}{5}+\dfrac{1.3.5}{2.4.6}\cdot\dfrac{x^7}{7}+\ldots$. [*Rajasthan*, '59]

23. $1+\frac{1}{2}x+\dfrac{2!}{3^2}x^2+\dfrac{3!}{4^3}x^3+\dfrac{4!}{5^4}x^4+\ldots$. [*U.P.C.S.*, 1960]

24. $\dfrac{a+x}{1}+\dfrac{(a+2x)^2}{2!}+\dfrac{(a+3x)^3}{3!}+\ldots$. [*Nagpur*, 1954]

25. $x^2(\log 2)^q+x^3(\log 3)^q+x^4(\log 4)^q+\ldots$. [*Lkw.*, '57]

26. Test for convergence

(*a*) $$\frac{1}{xy}-\frac{1}{(x+1)(y+1)}+\frac{1}{(x+2)(y+2)}-\frac{1}{(x+3)(y+3)}+\ldots.$$

(*b*) $$\frac{1}{x}+\frac{1}{x-1}+\frac{1}{x+1}+\frac{1}{x-2}+\frac{1}{x+2}+\ldots.$$

27. Test the convergence of $\sum \frac{U_n}{1-U_n}$ if ΣU_n is convergent and $U_n \neq 1$. [*Lucknow*, 1950]

28. If $\frac{u_n}{u_{n+1}} = \frac{n^k + An^{k-1} + Bn^{k-2} + Cn^{k-3} + \dots}{n^k + an^{k-1} + bn^{k-2} + cn^{k-3} + \dots}$,

where k is a positive integer, show that the series $u_1 + u_2 + u_3 + \dots$ is convergent if $A - a - 1$ is positive, and divergent if $A - a - 1$ is negative or zero. [*Allahabad*, 1943]

29. Show that the nth term tends to zero in the series

$$1 - \frac{1}{2} + \frac{1.3}{2.4} - \frac{1.3.5}{2.4.6} + \dots .$$

Hence prove that the series is convergent.

30. Show that the product

$$\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdots \frac{2n-1}{2n} \cdot \frac{2n+1}{2n}$$

tends to a finite limit as n tends to infinity.

CHAPTER VIII

DETERMINANTS

8·1. Definitions. A determinant is a particular type of expression written in a special concise form. Such expressions arise in the theory of linear equations.

Consider, for example, the equations

$$a_1x+b_1=0 \text{ and } a_2x+b_2=0.$$

The condition that they may be satisfied by the same value of x is $-b_1/a_1=-b_2/a_2$, i.e.,

$$a_1b_2-a_2b_1=0. \tag{1}$$

It is usual to denote the left-hand side of (1) by the symbol

$$\begin{vmatrix} a_1 & b_1 \\ a_1 & b_2 \end{vmatrix}.$$

This is called a *determinant of the second order*, and its value is $a_1b_2-a_2b_1$.

Let us now find the condition that the equations

$$a_1x+b_1y+c_1=0, \quad a_2x+b_2y+c_2=0,$$

and

$$a_3x+b_3y+c_3=0,$$

may be satisfied by the same set of values of x and y.

On solving the last two equations, we get

$$\frac{x}{b_2c_3-b_3c_2}=\frac{-y}{a_2c_3-a_3c_2}=\frac{1}{a_2b_3-a_3b_2}$$

or

$$\frac{x}{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}=\frac{-y}{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}}=\frac{1}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}.$$

If these values of x and y satisfy the first equation also, then

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0. \qquad (2)$$

We denote the left-hand side of this by

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \qquad (3)$$

and call (3) a *determinant of the third order.* Its value, as can be seen on expanding (2), is

$$a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2). \qquad (4)$$

Similarly, we call

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \qquad (5)$$

a determinant of the fourth order, its value being equal to

$$a_1\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix} - d_1\begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}. \qquad (6)$$

Each of the third order determinants can be expanded further to give an expression like (4).

In general, a determinant of the nth order contains n *rows* and n *columns*, and can be expressed in terms of determinants of the $(n-1)$th order in a way similar to the above. The members comprising the rows and columns of a determinant are called its *elements* or *constituents*.

The student should understand that a determinant is just an algebraic expression written in a convenient and concise form. The advantage of this form is apparent when we notice that the result of eliminating x and y between the equations

$$\left.\begin{array}{l} a_1x+b_1y+c_1=0 \\ a_2x+b_2y+c_2=0 \\ a_3x+b_3y+c_3=0 \end{array}\right\} \quad \text{is} \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0.$$

The elements in the determinant occur in the same order as in the given equations. This is true for a determinant of any order.

8·2. Expansion. (i) We have seen that a determinant of the nth order is equal to the sum of n determinants of $(n-1)$th order multiplied respectively by the elements of the first row taken alternately with plus and minus signs, the determinant of the $(n-1)$th order multiplying a particular element being obtained by omitting from the original determinant the column and the row containing that element.

By a repeated application of the above rule the value of any determinant can be finally obtained in a form free from determinants.

(ii) ALTERNATIVE METHOD. The value of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

is $\quad a_1b_2c_3-a_1b_3c_2+a_2b_3c_1-a_2b_1c_3+a_3b_1c_2-a_3b_2c_1.$

We notice that the determinant is equal to the sum of all the terms which can be obtained

by appending the suffixes 1, 2, 3 in all possible ways to a, b, c. The sign of the leading term $a_1b_2c_3$, in which the suffixes (and the letters too) occur in their natural order, is positive. The sign of any other term is positive or negative according as the number of interchanges required to bring the suffixes in their natural order is even or odd.

Thus the sign of $a_1b_3c_2$ is negative, for the order of the suffixes is 1, 3, 2 and one interchange is necessary to bring it to the order 1, 2, 3.

This method is applicable to determinants of any order.

8·3. Properties of Determinants. The properties which are proved below, though demonstrated only for third order determinants, can easily be shown to hold for determinants of any order.

(i) *The value of a determinant is not altered by changing its columns into rows and its rows into columns;* that is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \tag{1}$$

To prove this, we see that the determinant on the left

$$=a_1(b_2c_3-b_3c_2)-b_1(a_2c_3-a_3c_2)+c_1(a_2b_3-a_3b_2)$$
$$=a_1(b_2c_3-b_3c_2)-a_2(b_1c_3-c_1b_3)+a_3(b_1c_2-c_1b_2),$$

on rearrangement of the terms,

$=$the determinant on the right.

(ii) *If two adjacent rows or columns of a determinant are interchanged, the sign of the determinant is changed, but its numerical value is unaltered*, that is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = -\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix}. \tag{2}$$

To prove this, we notice that the right-hand side

$=-\{b_1(a_2c_3-a_3c_2)-a_1(b_2c_3-b_3c_2)+c_1(b_2a_3-b_3a_2)\}$

$=a_1(b_2c_3-b_3c_2)-b_1(a_2c_3-a_3c_2)+c_1(a_2b_3-a_3b_2)$

=the determinant on the left.

A similar result can be proved for an interchange of two rows.

NOTE. By successive applications of this property we can bring a row from one position to any other position. This shows that when we expand a determinant by the rule (i) of § 8·2, the determinant may be expanded in terms of the elements of any row, the row being brought to the top to determine the sign of the terms.

Also, by (i), we see that any result which we prove for the rows holds for the columns, and vice versa.

(iii) *If two rows or two columns of a determinant are identical, the value of the determinant is zero*; that is

$$\begin{vmatrix} a_1 & a_1 & c_1 \\ a_2 & a_2 & c_2 \\ a_3 & a_3 & c_3 \end{vmatrix} = 0. \tag{3}$$

Let Δ be the value of the determinant on the left. Then $-\Delta$ is the value of the determinant obtained by interchanging the first two columns. But as these columns are identical, there is no difference between the two determinants. Therefore

$$\Delta=-\Delta, \text{ i.e., } \Delta=0.$$

8·4. Minors and Cofactors. The determinant obtained from a given determinant Δ by omitting the row and the column which pass through any element is called the *minor* of that element in Δ.

Thus the minors of a_2, b_2, c_2 in the determinant

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

are respectively

$$\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}, \quad \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}, \quad \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix};$$

and in terms of these minors,

$$\Delta = -a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}.$$

It is, however, usually more convenient to write Δ in the form

$$\Delta = +a_2A_2 + b_2B_2 + c_2C_2$$

where

$$A_2 = -\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}, \; B_2 = \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}, \; C_2 = -\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}.$$

A_2, B_2, C_2 are then called the *cofactors* of a_2, b_2, c_2.

In terms of cofactors the result (3) of the preceding article can be written as

$$a_1B_1 + a_2B_2 + a_3B_3 = 0,$$

where B_1, B_2, B_3 are the cofactors of b_1, b_2, b_3 in Δ. Similarly it may be shown that

$$a_1A_2 + b_1B_2 + c_1C_2 = 0.$$

Thus we get the rule :

If the cofactors of the elements of a certain row (or column) are multiplied, in order, by the elements of another row (or column), the sum of the products is zero.

8·5. Further properties. (i) *If every element in a row, or in a column, of a determinant is multiplied by the same constant, then the value of the determinant gets multiplied by that constant;* that is

$$\begin{vmatrix} ka_1 & b_1 & c_1 \\ ka_2 & b_2 & c_2 \\ ka_3 & b_3 & c_3 \end{vmatrix} = k \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

Let Δ and Δ' denote respectively the determinants on the right and left, and let A_1, A_2, A_3 be the cofactors of a_1, a_2, a_3 in Δ. Then A_1, A_2, A_3 are also the cofactors of ka_1, ka_2, ka_3 in Δ'. Therefore

$$\Delta' = ka_1A_1 + ka_2A_2 + ka_3A_3$$
$$= k(a_1A_1 + a_2A_2 + a_3A_3) = k\Delta.$$

(ii) *If the elements in a row (or column) of a determinant are respectively equal to k times the corresponding elements in another row (or column), the value of the determinant is zero.*

For, on taking out the common factor k, the two rows (or columns) will become identical.

(iii) *To prove that*

$$\begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}.$$

Let D, Δ and Δ' denote the three determinants respectively; then the cofactors A_1, A_2, A_3 of the elements in the first column are the same for each determinant. Therefore

$$\begin{aligned} D &= (a_1+\alpha_1)A_1+(a_2+\alpha_2)A_2+(a_3+\alpha_3)A_3 \\ &= (a_1A_1+a_2A_2+a_3A_3)+(\alpha_1A_1+\alpha_2A_2+\alpha_3A_3) \\ &= \Delta+\Delta'. \end{aligned}$$

Similarly, if each element in a column contains m terms, the determinant can be expressed as the sum of m determinants.

We also see that

$$\begin{vmatrix} a_1+\alpha_1 & b_1+\beta_1 & c_1 \\ a_2+\alpha_2 & b_2+\beta_2 & c_2 \\ a_3+\alpha_3 & b_3+\beta_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1+\beta_1 & c_1 \\ a_2 & b_2+\beta_2 & c_2 \\ a_3 & b_3+\beta_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1+\beta_1 & c_1 \\ \alpha_2 & b_2+\beta_2 & c_2 \\ \alpha_3 & b_3+\beta_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & \beta_1 & c_1 \\ a_2 & \beta_2 & c_2 \\ a_3 & \beta_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & \beta_1 & c_1 \\ \alpha_2 & \beta_2 & c_2 \\ \alpha_3 & \beta_3 & c_3 \end{vmatrix}.$$

In general, if the elements in the three columns (or rows) contain m, n and p terms respectively, the determinant can be expressed as the sum of mnp determinants.

8·6. An Important Rule. *The value of a determinant is not altered if each element of a column (or row) is increased or diminished by the same multiple of the corresponding element of another column (or row);* that is,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+pb_1 & b_1 & c_1 \\ a_2+pb_2 & b_2 & c_2 \\ a_3+pb_3 & b_3 & c_3 \end{vmatrix}, \qquad (1)$$

where p may be positive or negative.

To prove this, we see that the determinant on the right

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} pb_1 & b_1 & c_1 \\ pb_2 & b_2 & c_2 \\ pb_3 & b_3 & c_3 \end{vmatrix} \qquad (2)$$

=the determinant on left-hand side of (1), since the second determinant in (2) is zero by (ii), § 8·5.

We can prove similarly that

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+pb_1+qc_1 & b_1 & c_1 \\ a_2+pb_2+qc_2 & b_2 & c_2 \\ a_3+pb_3+qc_3 & b_3 & c_3 \end{vmatrix},$$

and also

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+pb_1 & b_1+qc_1 & c_1 \\ a_2+pb_2 & b_2+qc_2 & c_2 \\ a_3+pb_3 & b_3+qc_3 & c_3 \end{vmatrix}.$$

Thus two (or more) additions and subtractions can be made in a single step. But care should be taken that one column (or row) is left unaltered, otherwise mistakes are likely to occur.

Ex. 1. Find the value of

$$\begin{vmatrix} 5 & 7 & 10 & 14 \\ 2 & 3 & 7 & 6 \\ 3 & 3 & 6 & 9 \\ 5 & 6 & 11 & 20 \end{vmatrix}.$$

As the second, third and fourth elements in the third row are multiples of the first element in this row, we shall try to make them zero. For this diminish each element of the second column by the corresponding element of the first column, diminish each element of the third column by

twice the corresponding element of the first column, and diminish each element of the fourth column by three times the corresponding element of the first column; then the given determinant

$$=\begin{vmatrix} 5 & 2 & 0 & -1 \\ 2 & 1 & 3 & 0 \\ 3 & 0 & 0 & 0 \\ 5 & 1 & 1 & 5 \end{vmatrix}.$$

Expanding this in terms of the elements of the third row we see that the determinant is equal to

$$3\begin{vmatrix} 2 & 0 & -1 \\ 1 & 3 & 0 \\ 1 & 1 & 5 \end{vmatrix}=3\begin{vmatrix} 0 & 0 & -1 \\ 1 & 3 & 0 \\ 11 & 1 & 5 \end{vmatrix},$$

on increasing the elements of the first column by twice those of the third. On expansion in terms of the elements of the first row, this gives

$$-3\begin{vmatrix} 1 & 2 \\ 11 & 1 \end{vmatrix}=-3(1-33)=96.$$

Hence the value of the given determinant is 96.

Ex. 2. Show that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}=(b-c)(c-a)(a-b)(a+b+c).$$

[*Delhi*, 1954]

Diminishing elements of the second and third columns by the corresponding elements of the first column, we see that the given determinant

$$=\begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^3 & b^3-a^3 & c^3-a^3 \end{vmatrix}=\begin{vmatrix} b-a & c-a \\ b^3-a^3 & c^3-a^3 \end{vmatrix}$$

$$=(b-a)(c-a)\begin{vmatrix} 1 & 1 \\ b^2+ab+a^2 & c^2+ac+a^2 \end{vmatrix}$$

$$=(b-a)(c-a)(c^2+ac-b^2-ab)$$

$$=(b-a)(c-a)(c-b)(c+b+a)$$
$$=(b-c)(c-a)(a-b)(a+b+c).$$

ALTERNATIVE METHOD. If we put $b=a$ in the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}, \qquad (1)$$

two of the columns become identical and the determinant vanishes. Therefore $a-b$ must be a factor of (1). Similarly, $b-c$ and $c-a$ are also factors of (1).

A consideration of the leading term bc^3 shows that the expression (1) must be of the fourth degree in a, b, c. Therefore, besides the factors $a-b$, $b-c$ and $c-a$, (1) will contain a linear factor. Also this factor will be symmetrical in a, b, c, as (1) is symmetrical in these. Hence

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)k(a+b+c).$$

A comparison of the term bc^3 on the two sides shows that $k=1$. Hence the result.

EXAMPLES

Calculate the values of the following determinants :

1. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$

2. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$

3. $\begin{vmatrix} 12 & 3 & 7 \\ 27 & 7 & 17 \\ 36 & 9 & 22 \end{vmatrix}.$

4. $\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}.$

[*Allahabad*, 1960]

5. $\begin{vmatrix} 21 & 17 & 7 & 10 \\ 24 & 22 & 6 & 10 \\ 6 & 8 & 2 & 3 \\ 5 & 7 & 1 & 2 \end{vmatrix}$.

[*Banaras*, 1949]

6. $\begin{vmatrix} 3 & 2 & 1 & 4 \\ 15 & 29 & 2 & 14 \\ 16 & 19 & 3 & 17 \\ 33 & 39 & 8 & 38 \end{vmatrix}$.

[*Agra*, 1958]

Prove the following identities :

7. $$\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}.$$ [*Utkal*, 1952]

8. $$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2.$$ [*Aligarh*, 1952]

9. $$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-c)(c-a)(a-b).$$

10. $$\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix} = -(ac-b^2)(ax^2+2bxy+cy^2).$$ [*Sagar*, 1949]

11. $$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3.$$ [*Kashmir*, 1951]

12. $$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = (y-z)(z-x)(x-y)(yz+zx+xy).$$ [*Rajasthan*, 1959]

13. $$\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix} = (a+b+c+d)(a-b+c-d) \times (a-b-c+d)(a+b-c-d).$$ [*Nagpur*, 1950]

Solve the following equations :

14. $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0.$ [*Agra*, 1951]

15. $\begin{vmatrix} 4x & 6x+2 & 8x+1 \\ 6x+2 & 9x+2 & 12x \\ 8x+1 & 12x & 16x+2 \end{vmatrix} = 0.$ [*Lucknow*, 1950]

8·7. Multiplication of determinants.

$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } \Delta' \equiv \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix},$$

then $\Delta\Delta' = D$, where $D \equiv$

$$\begin{vmatrix} a_1\alpha_1+b_1\beta_1+c_1\gamma_1 & a_1\alpha_2+b_1\beta_2+c_1\gamma_2 & a_1\alpha_3+b_1\beta_3+c_1\gamma_3 \\ a_2\alpha_1+b_2\beta_1+c_2\gamma_1 & a_2\alpha_2+b_2\beta_2+c_2\gamma_2 & a_2\alpha_3+b_2\beta_3+c_2\gamma_3 \\ a_3\alpha_1+b_3\beta_1+c_3\gamma_1 & a_3\alpha_2+b_3\beta_2+c_3\gamma_2 & a_3\alpha_3+b_3\beta_3+c_3\gamma_3 \end{vmatrix}.$$

To prove this we see that the determinant D can be expressed as the sum of $3\times3\times3$, i.e., 27 determinants.

One of these determinants is

$$\begin{vmatrix} a_1\alpha_1 & a_1\alpha_2 & b_1\beta_3 \\ a_2\alpha_1 & a_2\alpha_2 & b_2\beta_3 \\ a_3\alpha_1 & a_3\alpha_2 & b_3\beta_3 \end{vmatrix} = \alpha_1\alpha_2\beta_3 \begin{vmatrix} a_1 & a_1 & b_1 \\ a_2 & a_2 & b_2 \\ a_3 & a_3 & b_3 \end{vmatrix},$$

which is zero, because two of the columns are identical.

A little consideration will show that 21 out of the 27 determinants will be zero. Of the determinants which do not vanish, one is

$$\begin{vmatrix} a_1\alpha_1 & b_1\beta_2 & c_1\gamma_3 \\ a_2\alpha_1 & b_2\beta_2 & c_2\gamma_3 \\ a_3\alpha_1 & b_3\beta_2 & c_3\gamma_3 \end{vmatrix} = \alpha_1\beta_2\gamma_3 \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \alpha_1\beta_2\gamma_3\Delta.$$

Similarly the other non-zero determinants contain Δ as a factor. Collecting them together, we see that

$$D = (\alpha_1\beta_2\gamma_3 - \alpha_1\beta_3\gamma_2 + \alpha_2\beta_3\gamma_1 - \alpha_2\beta_1\gamma_3 + \alpha_3\beta_1\gamma_2 - \alpha_3\beta_2\gamma_1)\Delta$$

$$= \Delta' . \Delta.$$

A similar result holds for determinants of any order.

NOTE. Since we can interchange two rows, or rows into columns in one or both of the determinants Δ and Δ' before multiplication, the product of two determinants can be expressed as a determinant in several ways. But on expansion these will all give the same result.

Sometimes the student is required to express a given determinant as a product of two determinants. In such cases he should try to reconstruct Δ and Δ' by an inspection of the terms in D, as has been done in the example below.

Ex. Express $\begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix}$

as a product of two determinants. [*Banaras*, 1960]

The given determinant, D say,

$$= \begin{vmatrix} a^2-2ax+x^2 & b^2-2bx+x^2 & c^2-2cx+x^2 \\ a^2-2ay+y^2 & b^2-2by+y^2 & c^2-2cy+y^2 \\ a^2-2az+z^2 & b^2-2bz+z^2 & c^2-2cz+z^2 \end{vmatrix}. \quad (1)$$

If this is equal to $\Delta\Delta'$, then an inspection of the elements shows that the three rows in Δ will contain terms in x, y and z respectively; and the rows in Δ' will contain terms in a, b, c respectively.

Also the first element in the first row can be written as

$$1 . a^2+x(-2a)+x^2 . 1 ;$$

therefore let us put, tentatively,

$$D=\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \times \begin{vmatrix} a^2 & -2a & 1 \\ b^2 & -2b & 1 \\ c^2 & -2c & 1 \end{vmatrix}.$$

That this is the correct result can be seen by multiplying out the last two determinants and comparing the result with (1).

8·71. Product of two determinants. Alternative proof. Consider the equations

$$\left.\begin{aligned} a_1X+b_1Y=0 \\ a_2X+b_2Y=0 \end{aligned}\right\}, \qquad . \quad . \quad . \quad (1)$$

where

$$\left.\begin{aligned} X=\alpha_1x+\alpha_2y \\ Y=\beta_1x+\beta_2y \end{aligned}\right\} \qquad . \quad . \quad . \quad (2)$$

and suppose that x and y are not zero.

Then, in order that equations (1) may hold simultaneously we must have either

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=0, \qquad (3)$$

or $X=0, Y=0$. This latter condition gives, by (2),

$$\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix}=0. \qquad (4)$$

But if in (1) we first substitute for X and Y their values in terms of x and y, we get

$$\left.\begin{aligned} (a_1\alpha_1+b_1\beta_1)x+(a_1\alpha_2+b_1\beta_2)y=0 \\ (a_2\alpha_1+b_2\beta_1)x+(a_2\alpha_2+b_2\beta_2)y=0 \end{aligned}\right\}. \qquad (5)$$

Now the condition that the equations (5) may hold simultaneously is that

$$\begin{vmatrix} a_1\alpha_1+b_1\beta_1 & a_1\alpha_2+b_1\beta_2 \\ a_2\alpha_1+b_2\beta_1 & a_2\alpha_2+b_2\beta_2 \end{vmatrix}=0. \qquad (6)$$

This condition must be equivalent to the conditions (3) and (4) and will be satisfied if either of (3) and (4) is satisfied. Therefore, the determinant in (5) must contain the

determinants occurring in (3) and (4) as factors. A consideration of the dimensions shows that the remaining factor must be numerical, and comparison of the coefficients of $a_1b_2\alpha_1\beta_2$ shows that this factor is unity. Hence

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = \begin{vmatrix} a_1\alpha_1+b_1\beta_1 & a_1\alpha_2+b_1\beta_2 \\ a_2\alpha_1+b_2\beta_1 & a_2\alpha_2+b_2\beta_2 \end{vmatrix}.$$

It is evident that the above method is perfectly general and applicable to the product of two determinants of any order.

8·8. Solution of Simultaneous Equations. The properties of determinants can be used to solve linear simultaneous equations.

Suppose we have to solve the equations

$$a_1x+b_1y+c_1z+d_1=0,$$
$$a_2x+b_2y+c_2z+d_2=0,$$
$$a_3x+b_3y+c_3z+d_3=0.$$

Let
$$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

and denote the cofactors of the elements in Δ by the corresponding capital letters.

Multiply the given equations by A_1, A_2, A_3 respectively and add; then we get

$$(a_1A_1+a_2A_2+a_3A_3)x+(d_1A_1+d_2A_2+d_3A_3)=0,$$

the other terms vanishing by § 8·4. As the coefficient of x is equal to Δ, we see that

$$x=-(d_1A_1+d_2A_2+d_3A_3)/\Delta.$$

Similar results hold for y and z.

Expressing the numerators $d_1A_1+d_2A_2+d_3A_3$, etc., which occur in the values of x, y and z, as determinants, we see that

$$\frac{x}{\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

EXAMPLES ON CHAPTER VIII

Find the values of the following determinants :

1. $\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$ [*Lucknow*, 1954]

2. $\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 69 \end{vmatrix}$. [*Allahabad*, 1951]

3. $\begin{vmatrix} a & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix}$. [*Annam.*, 1953]

4. $\begin{vmatrix} 0 & a & b & c \\ a & 0 & c & b \\ b & c & 0 & a \\ c & b & a & 0 \end{vmatrix}$. [*Andhra*, 1960]

5. $\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix}$. [*Rajasthan*, 1957]

6. $\begin{vmatrix} a^3 & a^2 & a & 1 \\ b^3 & b^2 & b & 1 \\ c^3 & c^2 & c & 1 \\ d^3 & d^2 & d & 1 \end{vmatrix}$. [*U.P.C.S.*, 1947]

7. $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & 1 & \omega \end{vmatrix}$, where ω is one of the imaginary cube roots of unity [*Delhi*, 1958]

8. Prove that $\begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$,

and find the value of the second determinant. [*Aligarh*, 1953]

9. If $a+b+c=0$, solve the equation

$$\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0.$$

[*Andhra*, 1954]

10. Show that $x=2$ is a root of the equation

$$\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix}=0,$$

and solve it completely. [*Agra*, 1948]

Prove the following identities :

11. $$\begin{vmatrix} 1 & bc+ad & b^2c^2+a^2d^2 \\ 1 & ca+bd & c^2a^2+b^2d^2 \\ 1 & ab+cd & a^2b^2+c^2d^2 \end{vmatrix} =-(b-c)(c-a)(a-b)(a-d)(b-d)(c-d).$$

12. $$\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix} =(b-c)(c-a)(a-b)(a+b+c)(a^2+b^2+c^2).$$

[*Allahabad*, 1955]

13. $$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} =2abc(a+b+c)^3.$$

[*Gauhati*, 1959]

14. $$\begin{vmatrix} (a-x)^2 & (b-x)^2 & (c-x)^2 \\ (a-y)^2 & (b-y)^2 & (c-y)^2 \\ (a-z)^2 & (b-z)^2 & (c-z)^2 \end{vmatrix} =2(b-c)(c-a)(a-b)(y-z)(z-x)(x-y).$$

[*Nagpur*, 1954]

15. $$\begin{vmatrix} 4 & 5 & 6 & x \\ 5 & 6 & 7 & y \\ 6 & 7 & 8 & z \\ x & y & z & 0 \end{vmatrix} =(x-2y+z)^2.$$

[*Lucknow*, 1955]

16. $$\begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b & b^2 & b^3+cda \\ 1 & c & c^2 & c^3+dab \\ 1 & d & d^2 & d^3+abc \end{vmatrix} =0.$$

[*Gorakhpur*, 1959]

17. $$\begin{vmatrix} a^2+1 & ab & ac & ad \\ ba & b^2+1 & bc & bd \\ ca & cb & c^2+1 & cd \\ da & db & dc & d^2+1 \end{vmatrix} = a^2+b^2+c^2+d^2+1.$$

[*Karnatak*, 1954]

18. $$\begin{vmatrix} b^2+c^2+1 & c^2+1 & b^2+1 & b+c \\ c^2+1 & c^2+a^2+1 & a^2+1 & c+a \\ b^2+1 & a^2+1 & a^2+b^2+1 & a+b \\ b+c & c+a & a+b & 3 \end{vmatrix} = (bc+ca+ab)^2$$

[*Raj.*, 1950]

19. Express as a determinant $\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2.$

20. If ω is one of the imaginary cube roots of unity show that

$$\begin{vmatrix} 1 & \omega & \omega^2 & \omega^3 \\ \omega & \omega^2 & \omega^3 & 1 \\ \omega^2 & \omega^3 & 1 & \omega \\ \omega^3 & 1 & \omega & \omega^2 \end{vmatrix}^2 = \begin{vmatrix} 1 & 1 & -2 & 1 \\ 1 & 1 & 1 & -2 \\ -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \end{vmatrix};$$

and hence show that the value of the determinant on the left is $3\sqrt{(-3)}$.

[*U.P.F.S.*, 1960]

21. Prove that the determinant

$$\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$$

is a perfect square, and find its value. [*Aligarh*, 1953]

22. Express

$$\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$$

as a product of two determinants and hence find its value.

[*Gorakhpur*, 1960]

23. If $A_1, B_1, C_1, \ldots$ are the cofactors of $a_1, b_1, c_1, \ldots$ in

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

then show that
$$\Delta^2 = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}.$$

[*Osmania*, '54]

24. Show that

$$\begin{vmatrix} yz-x^2 & zx-y^2 & xy-z^2 \\ zx-y^2 & xy-z^2 & yz-x^2 \\ xy-z^2 & yz-x^2 & zx-y^2 \end{vmatrix} = \begin{vmatrix} r^2 & u^2 & u^2 \\ u^2 & r^2 & u^2 \\ u^2 & u^2 & r^2 \end{vmatrix},$$

where $r^2 = x^2 + y^2 + z^2$ and $u^2 = yz + zx + xy$.

CHAPTER IX

MATRICES

9·1. Definition. *A set of mn numbers arranged in an array of m rows and n columns,*

$$\begin{bmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ b_1 & b_2 & b_3 & \dots & b_n \\ \dots & \dots & \dots & \dots & \dots \\ k_1 & k_2 & k_3 & \dots & k_n \end{bmatrix},$$

is called a MATRIX.

When $m=n$ the array is called a square matrix of order n.

A matrix is frequently denoted by a single letter A. The numbers a_1, a_2, ... are called the *elements* of the matrix.

In elementary algebra attention is mainly focussed on single numbers. These numbers are combined by various operations, such as addition, multiplication, etc., to obtain other numbers. But in some branches of algebra one has to consider a set of numbers. For example, in analytical geometry of three dimensions, the coordinates of a point are given by a set of three numbers; and in a case of transformation of coordinate axes, the direction cosines of the new axes are given by a set of nine numbers : three for each axis. The former set of numbers could be represented by the matrix $[x\ y\ z]$, and the latter by the square matrix

$$\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}.$$

When operating with such a set of numbers, it is often more convenient to treat the set as a single entity. For example, the set of simultaneous equations

$$a_1x_1+\ a_2x_2+\ a_3x_3=b_1,$$
$$a'_1x_1+a'_2x_2+a'_3x_3=b_2,$$
$$a''_1x_1+a''_2x_2+a''_3x_3=b_3,$$

may be symbolically represented by the simple equation

$$AX=B.$$

Here A will represent the array of numbers $a_1, a_2, \ldots,$ X the numbers x_1, x_2, x_3 and B the numbers b_1, b_2, b_3. Unlike a determinant, a matrix cannot reduce to a single number, and the problem of determining the 'value' of a matrix never arises.

But if we give suitable definitions for addition, multiplication, etc., of matrices, we obtain a powerful and convenient method for dealing with problems connected with sets of simultaneous equations. These definitions are given in the sections which follow.

The theory of matrices has been found of great utility in many branches of higher mathematics, such as algebraic and differential equations, astronomy, mechanics, the theory of electrical circuits, quantum mechanics, nuclear physics and aerodynamics. In the elementary treatment given here only the application of matrices to the solution of algebraic linear simultaneous equations is considered.

To indicate the position of an element in a matrix, the elements are usually denoted by a letter followed by two suffixes. Thus a matrix with m rows and n columns may be written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}.$$

The suffixes r and s in the element a_{rs} indicate that the element a_{rs} is in the rth row and sth column. In this notation the symbol $[a_{rs}]$ is generally used to denote the above matrix.

9·2. Addition of matrices and other definitions. (*i*) *Two matrices A and B are said to be equal if A has the same number of rows and columns as B, and each element of A is equal to the corresponding element of B.* Thus, for equality, the two matrices have to be identical in every respect.

(*ii*) *If two matrices A and B have the same number of rows and the same number of columns, then the sum of A and B is defined as the matrix each element of which is the sum of the corresponding elements of A and B.* Thus

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} + \begin{bmatrix} c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{bmatrix} = \begin{bmatrix} a_1+c_1 & a_2+c_2 & a_3+c_3 \\ b_1+d_1 & b_2+d_2 & b_3+d_3 \end{bmatrix}.$$

Similarly, $A-B$ is defined as the matrix whose elements are obtained by subtracting the elements of B from the corresponding elements of A, e.g.

$$\begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 2 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

(*iii*) *If k is a number and A a matrix, then kA is defined as the matrix each element of which is k times the corresponding element of A*, that is,

$$k\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} = \begin{bmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \end{bmatrix}.$$

Since the addition and subtraction of matrices are based directly on the addition and subtraction of their elements, it follows that the commutative and associative laws, viz.

$$A+B=B+A,$$

and

$$(A+B)+C=A+(B+C),$$

will hold good for the addition and subtraction of matrices also. Similarly, the distributive law

$$k(A+B)=kA+kB$$

holds when k is a number and A and B are matrices.

It also follows from (ii) and (iii) that we may write $2A$ for $A+A$, $4A$ for $7A-3A$, and so on. In other words, the matrices obey the ordinary laws of algebra.

Ex. If $A=\begin{bmatrix}2 & 3 & 1\\ 0 & -1 & 5\end{bmatrix}$ and $B=\begin{bmatrix}1 & 2 & -1\\ 0 & -1 & 3\end{bmatrix}$, find $2A-3B$.

$$2A-3B=\begin{bmatrix}4 & 6 & 2\\ 0 & -2 & 10\end{bmatrix}-\begin{bmatrix}3 & 6 & -3\\ 0 & -3 & 9\end{bmatrix}=\begin{bmatrix}1 & 0 & 5\\ 0 & 1 & 1\end{bmatrix}.$$

9·3. Multiplication of matrices. The product AB of two matrices A and B, is defined only when the number of columns in A is equal to the number of rows in B, and follows a pattern somewhat similar to that for the product of two determinants. For example, the product

$$\begin{bmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\\ a_4 & b_4 & c_4\end{bmatrix}\times\begin{bmatrix}\alpha_1 & \alpha_2\\ \beta_1 & \beta_2\\ \gamma_1 & \gamma_2\end{bmatrix} \quad . \; . \; . \; (1)$$

is defined as the matrix

$$\begin{bmatrix}a_1\alpha_1+b_1\beta_1+c_1\gamma_1 & a_1\alpha_2+b_1\beta_2+c_1\gamma_2\\ a_2\alpha_1+b_2\beta_1+c_2\gamma_1 & a_2\alpha_2+b_2\beta_2+c_2\gamma_2\\ a_3\alpha_1+b_3\beta_1+c_3\gamma_1 & a_3\alpha_2+b_3\beta_2+c_3\gamma_2\\ a_4\alpha_1+b_4\beta_1+c_4\gamma_1 & a_4\alpha_2+b_4\beta_2+c_4\gamma_2\end{bmatrix}. \qquad (2)$$

To get the product (2) we take one row from the first matrix and one column from the second matrix in (1), and obtain the product of each element in the row with the corresponding element in the column. The sum of these products forms an element of the product matrix (2).

If we start with the rth row of the first matrix and the sth column of the second matrix, we obtain the element in the rth row and sth column of the product matrix.

Using elements with double suffixes, the rule for multiplication of matrices can be stated as follows.

Let A and B be the matrices

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & b_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1} & b_{n2} & \cdots & b_{np} \end{bmatrix},$$

with m rows and n columns, and n rows and p columns, respectively. Then the product AB is defined as the matrix whose element in the rth row and sth column is

$$a_{r1}b_{1s}+a_{r2}b_{2s}+a_{r3}b_{3s}+\cdots+a_{rn}b_{ns}.$$

The product matrix will have m rows and p columns.

The student should note that the addition or multiplication of two matrices A and B have been defined under certain restrictions. A and B can be added only when A has the same number of rows and columns as B, while the product AB can be formed only when the number of columns in A is equal to the number of rows in B. We express this by saying respectively that A and B are *conformable* for addition, or conformable for the product AB. The student should also note that two matrices A and B may not be conformable for both the products AB and BA, and even if they are, $AB \neq BA$, in general.

When A is a square matrix we can form the product AA, i.e., A^2. We can also form the continued product ABC of the matrices A, B and C, provided A and B are conformable for the product AB, and B and C are conformable for the product BC. In such a case, it can be shown that $(AB)C = A(BC)$, and no ambiguity arises when we write ABC.

Ex. If $A=\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$, (i) form the product AB, and (ii) show that $A^3=4A$.

(i) $$AB=\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\times\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 1-3 & 2-2 & 3-1 \\ -1+3 & -2+2 & -3+1 \end{bmatrix}=\begin{bmatrix} -2 & 0 & 2 \\ 2 & 0 & -2 \end{bmatrix}.$$

(ii) $$A^2=\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\times\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}=\begin{bmatrix} 1+1 & -1-1 \\ -1-1 & 1+1 \end{bmatrix}$$

$$=\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}=2A.$$

Therefore $$A^3=A^2A=2AA=2A^2=4A.$$

9·4. Special matrices. We have already defined a *square* matrix as a matrix in which the number of rows is equal to the number of columns. A *row* matrix is defined as a matrix having a single row, e.g., [1 2 3]. A *column* matrix is a matrix having a single column, e.g.,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

A square matrix of order n which has unity for its elements in the leading diagonal and zero elsewhere is called the *unit matrix* of order n. It is generally denoted by I.

A matrix having every element zero is called a *null matrix*. It is denoted by 0 or O.

Examples of unit and null matrices are respectively

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The student can easily verify that if A be a square matrix of the same order as I,

$$IA=AI=A.$$

Also $$I=I^2=I^3=\dots.$$

If A is not a square matrix but has m rows and n columns even then $IA=A$, and $AI=A$, provided that in IA the unit matrix is of order m while in AI it is of order n.

Similarly, it can be verified that

$$OA=AO=O,$$

where O is a null matrix. One should note, however, that $AB=0$ does not necessarily imply that A or B is a null matrix. For instance, if

$$A=\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \text{ and } B=\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

the product AB is a null matrix, although neither A nor B is a null matrix.

9·5. Related Matrices. A matrix obtained from a given matrix A by changing its rows into columns (and vice versa) is called the *transposed matrix* of A. Thus the matrix

$$\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}$$

is the transposed matrix of

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_1 \end{bmatrix}.$$

The transposed matrix of A is usually denoted by A'.

When we have a square matrix

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \qquad \ldots \quad (1)$$

we can form a determinant Δ from the elements of the matrix. If $\Delta=0$, we call the given matrix a *singular matrix*. If $\Delta\neq0$, it is an ordinary or non-singular matrix.

The cofactors $A_1, B_1, C_1, \dots$ of the elements $a_1, b_1, c_1, \dots$ in Δ can be used to form a matrix. The transpose of this matrix, namely

$$\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \quad . \quad . \quad . \quad (2)$$

is called the *adjoint* matrix of (1).

The matrix obtained by dividing each element of (2) by Δ is called the *reciprocal* matrix of (1). Evidently the reciprocal matrix is defined only when the original matrix is non-singular. If the original matrix is denoted by A, its adjoint is denoted by adj A and its reciprocal by A^{-1}. Also, from definition, $A^{-1} = (\text{adj } A)/\Delta$.

A property of the reciprocal matrix is that

$$AA^{-1} = A^{-1}A = I.$$

To prove it we see that

$$AA^{-1} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \times \frac{1}{\Delta} \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

$$= \frac{1}{\Delta} \begin{bmatrix} a_1A_1 + b_1B_1 + c_1C_1 & a_1A_2 + b_1B_2 + c_1C_2 & a_1A_3 + b_1B_3 + c_1C_3 \\ a_2A_1 + b_2B_1 + c_2C_1 & a_2A_2 + b_2B_2 + c_2C_2 & a_2A_3 + b_2B_3 + c_2C_3 \\ a_3A_1 + b_3B_1 + c_3C_1 & a_3A_2 + b_3B_2 + c_3C_2 & a_3A_3 + b_3B_3 + c_3C_3 \end{bmatrix}$$

$$= \frac{1}{\Delta} \begin{bmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We may similarly prove that $A^{-1}A = I$.

9·6. Solution of simultaneous equations. Keeping in mind the rules for multiplication of matrices and their equality, the student will easily see that the equations

$$\left. \begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned} \right\} \qquad (1)$$

are equivalent to the matrix equation

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}, \qquad (2)$$

or, briefly by,

$$AX=D, \tag{3}$$

where A, X and D represent respectively the three matrices in (2). Multiplying both sides of (3) by the reciprocal matrix A^{-1}, we get

$$A^{-1}AX=A^{-1}D,$$

or

$$IX=A^{-1}D,$$

or

$$X=A^{-1}D,$$

i.e.,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\frac{1}{\Delta}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}\times\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}.$$

This gives the solution of equations (1).

Ex. Solve, with the help of matrices, the simultaneous equations

$$x+y+z=3,$$
$$x+2y+3z=4,$$
$$x+4y+9z=6.$$

Here $\Delta=\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix}=\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 6 \end{vmatrix}=2.$

$$A_1=\begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix}=6,\ B_1=\begin{vmatrix} 3 & 1 \\ 9 & 1 \end{vmatrix}=-6,\ C_1=\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}=2,$$

$$A_2=\begin{vmatrix} 4 & 9 \\ 1 & 1 \end{vmatrix}=-5,\ B_2=\begin{vmatrix} 9 & 1 \\ 1 & 1 \end{vmatrix}=8,\ C_2=\begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix}=-3,$$

$$A_3=\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}=1,\ B_3=\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}=-2,\ C_3=\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}=1.$$

Therefore $\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\frac{1}{2}\begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}\times\begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}=\frac{1}{2}\begin{bmatrix} 18-20+6 \\ -18+32-12 \\ 6-12+6 \end{bmatrix}$

$$=\frac{1}{2}\begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}.$$

So $\qquad x=2,\ y=1,\ z=0.$

Examples on Chapter IX

1. Evaluate

(i) $[1 \quad 2 \quad 3]+[4 \quad 5 \quad 6]$.

(ii) $\begin{bmatrix} 1 & 4 \\ 2 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix} - \begin{bmatrix} 3 & 8 \\ -2 & 5 \end{bmatrix}$.

(iii) $3\begin{bmatrix} 1 & 6 & 2 \\ 4 & 3 & -5 \end{bmatrix} -2 \begin{bmatrix} 2 & 9 & -6 \\ 4 & -5 & 3 \end{bmatrix}$.

2. If $A=\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}$,

form the product AB. Is BA defined ?

3. If $A=\begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$ and $B=\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$,

form the products AB and BA, and show that $AB \neq BA$.

4. Form the product, AB and BA, when

(i) $A=\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B=\begin{bmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$.

(ii) $A=[1 \quad 2 \quad 3 \quad 4]$, $B=\begin{bmatrix} 5 \\ 4 \\ 3 \\ 2 \end{bmatrix}$.

5. If A denotes the first matrix of Ex. 3, and I the unit matrix of order 3, evaluate $A^2-3A+9I$.

6. If A denotes the first matrix of Ex. 3, obtain its transpose A' and form the product AA'.

7. Evaluate

(i) $\begin{bmatrix} 2 & 1 & -1 \\ 4 & -5 & 6 \\ -3 & 7 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 3 \\ 4 & -6 \\ -2 & 5 \end{bmatrix} \times \begin{bmatrix} 5 & 3 \\ -2 & 1 \end{bmatrix}$.

(ii) $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \times [4 \quad 5 \quad 2] \times \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} \times [3 \quad 2]$.

8. If $A=\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$, evaluate A^2, A^3 and A^{-1}.

9. If $A=\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$,

show that $A^n=\begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$,

when n is a positive integer.

[Hint. Apply the method of induction.]

10. Prove that

$$[x\ y\ z].\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}.\begin{bmatrix} x \\ y \\ z \end{bmatrix}=[ax^2+by^2+cz^2+2fyz+2gzx+2hxy].$$

[*Rajasthan*, 1960]

11. Obtain the matrices A', adj A and A^{-1}, when

$$A=\begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & -6 & -7 \end{bmatrix}.$$

12. If e^A is defined as $I+A+A^2/2!+A^3/3!+\ldots$, show that $e^A=e^x\begin{bmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{bmatrix}$, when $A=\begin{bmatrix} x & x \\ x & x \end{bmatrix}$.

CHAPTER X

SUMMATION OF SERIES

10·1. Method of Differences. We have considered in chapters I and II the summation of binomial and exponential series, and in chapter V the summation of recurring series. The student will already be familiar with the summation of arithmetic and geometric series.

We shall now consider the summation of a series

$$u_1 + u_2 + u_3 + \ldots + u_n$$

by expressing its general term u_r as a difference of two terms $v_r - v_{r-1}$. If we are able to do so, the sum will be

$$(v_1 - v_0) + (v_2 - v_1) + (v_3 - v_2) + \ldots + (v_n - v_{n-1}),$$

i.e.,

$$v_n - v_0.$$

10·2. u_n a product of r factors in A.P. *To find the sum to n terms of the series for which*

$$u_n = \{a + nb\}\{a + (n+1)b\} \ldots \{a + (n+r-1)b\},$$

where a, b, r are constants.

Here every term is the product of r factors in arithmetical progression, and the first factors of the terms $u_1, u_2, \ldots,$ are also in arithmetical progression with the same common difference. We notice that

$$u_n = \frac{u_n[\{a + (n+r)b\} - \{a + (n-1)b\}]}{(r+1)b}$$

$$= \frac{u_n\{a + (n+r)b\}}{(r+1)b} - \frac{\{a + (n-1)b\}u_n}{(r+1)b}. \qquad (1)$$

Writing $n-1$ for n in the numerator of the first term gives us $u_{n-1}\{a+(n+r-1)b\}$, which is easily seen, on writing out the factors of u_{n-1} in full, to be the same as the numerator of the second term in (1). Moreover, the denominators are the same. Therefore, if we denote the first term by v_n, the second term will be v_{n-1}. Hence

$$u_n = v_n - v_{n-1},$$

where
$$v_n = \frac{u_n\{a+(n+r)b\}}{(r+1)b},$$

and, by § 10·1, the sum to n terms

$$= v_n - v_0.$$

Since v_0 is independent of n, we get the following rule for the sum to n terms :

Multiply u_n by the next factor in the A.P., divide by the number of factors thus increased and by the common difference, and add a constant.

The constant is found by putting $n=1$ in the result, or by putting $n=0$ in v_n, as below.

Ex. 1. Find the sum to n terms of the series

$$1 \,.\, 3 \,.\, 5 + 3 \,.\, 5 \,.\, 7 + 5 \,.\, 7 \,.\, 9 + \dots .$$

Here $u_n = (2n-1)(2n+1)(2n+3)$; hence by the rule

$$s_n = \frac{(2n-1)(2n+1)(2n+3)(2n+5)}{4 \,.\, 2} + C.$$

To determine C, put $n=1$. Since $s_1 = u_1 = 1 \,.\, 3 \,.\, 5$, we get

$$1 \,.\, 3 \,.\, 5 = \tfrac{1}{8}\{1 \,.\, 3 \,.\, 5 \,.\, 7\} + C, \quad \text{or} \quad C = \tfrac{15}{8}.$$

Therefore $\quad s_n = \tfrac{1}{8}\{(2n-1)(2n+1)(2n+3)(2n+5)+15\}.$

Ex. 2. Sum to n terms the series

$$1 \,.\, 2 \,.\, 3 + 2 \,.\, 3 \,.\, 5 + 3 \,.\, 4 \,.\, 7 + \dots .$$

Here $u_n = n(n+1)(2n+1)$, and is not of the standard form. But we can reduce it as follows.

$$\begin{aligned} u_n &= n(n+1)(2n+4-3) \\ &= 2 \,.\, n(n+1)(n+2) - 3 \,.\, n(n+1), \end{aligned}$$

both terms now being of the standard form. Therefore by the rule,

$$s_n = 2n(n+1)(n+2)(n+3)/4 - 3n(n+1)(n+2)/3,$$

the constant being zero as $u_0 = 0$ for both the terms,

$$= \tfrac{1}{2}n(n+1)(n+2)\{n+3-2\}$$
$$= \tfrac{1}{2}n(n+1)^2(n+2).$$

10·3. v_n the reciprocal of r factors in A.P.

To find the sum to n terms of the series for which

$$u_n = 1/[\{a+nb\}\{a+(n+1)b\}\dots\{a+(n+r-1)b\}],$$

where a, b and r are constants.

If we take

$$v_n = 1/[\{a+(n+1)b\}\{a+(n+2)b\}\dots\{a+(n+r-1)b\}],$$

then $$v_{n-1} - v_n = \frac{\{a+(n+r-1)b\} - \{a+nb\}}{\{a+nb\}\{a+(n+1)b\}\dots\{a+(n+r-1)b\}}$$
$$= (r-1)b\,.\,u_n.$$

Therefore $u_1 + u_2 + \dots + u_n$

$$= \frac{(v_0 - v_1) + (v_1 - v_2) + \dots + (v_{n-1} - v_n)}{(r-1)b}$$
$$= (v_0 - v_n)/(r-1)b.$$

Since $v_0/(r-1)b$ is a constant we get the following rule for the sum.

Remove the first factor from the denominator of u_n, divide by the number of factors thus diminished and by the common difference of the A.P., and subtract the result from a constant.

The constant is found by putting $n=1$ in the result, or by finding v_0.

Ex. 1. Sum to n terms the series

$$\frac{4}{2\,.\,3\,.\,4} + \frac{7}{3\,.\,4\,.\,5} + \frac{10}{4\,.\,5\,.\,6} + \dots \qquad \text{[Annamalai, '53]}$$

Here $$u_n = \frac{3n+1}{(n+1)(n+2)(n+3)} = \frac{3n+3-2}{(n+1)(n+2)(n+3)}$$
$$= \frac{3}{(n+2)(n+3)} - \frac{2}{(n+1)(n+2)(n+3)}.$$

Hence, by the rule,
$$s_n = C - \frac{3}{n+3} + \frac{2}{2(n+2)(n+3)} = C - \frac{3n+5}{(n+2)(n+3)}.$$

Putting $n=1$, we get
$$\frac{4}{2 \,.\, 3 \,.\, 4} = C - \frac{8}{3 \,.\, 4}, \quad \text{or} \quad C = \frac{5}{6}.$$

Therefore $s_n = \frac{5}{6} - (3n+5)/(n+2)(n+3)$.

Ex. 2. Show that
$$\frac{1}{1 \,.\, 2 \,.\, 3} + \frac{1}{3 \,.\, 4 \,.\, 5} + \frac{1}{5 \,.\, 6 \,.\, 7} + \dots \text{ to } \infty = \log 2 - \tfrac{1}{2}.$$

[*Poona*, 1960]

Here $u_n = 1/\{(2n-1)(2n)(2n+1)\}$, and though the factors in the denominator are in A.P., the first factors of the terms u_1, u_2, ... are not in the same A.P. Therefore the method of the preceding article is not applicable. Breaking up into partial fractions, we see that
$$u_n = \frac{1}{2(2n-1)} - \frac{1}{n} + \frac{1}{2(2n+1)}.$$

Hence, the given series
$$= \frac{1}{2}\left\{\left(1 - \frac{2}{2} + \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{2}{6} + \frac{1}{7}\right) + \dots\right\}$$
$$= \tfrac{1}{2} - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \dots = \log 2 - \tfrac{1}{2}.$$

10·4. A third type. *To find the sum to n terms of the series*
$$\frac{a}{b} + \frac{a(a+x)}{b(b+x)} + \frac{a(a+x)(a+2x)}{b(b+x)(b+2x)} + \dots.$$

Here $$u_n = \frac{a(a+x)(a+2x)\dots\{a+(n-1)x\}}{b(b+x)(b+2x)\dots\{b+(n-1)x\}}.$$

Let $v_n = u_n(a+nx)$.

Then $v_{n-1}=u_{n-1}\{a+(n-1)x\}$

$$=\frac{a(a+x)\ldots\{a+(n-2)x\}\{a+(n-1)x\}}{b(b+x)\ldots\{b+(n-2)x\}}$$

$$=u_n\{b+(n-1)x\},$$

so that $v_n-v_{n-1}=u_n[(a+nx)-\{b+(n-1)x\}]$

$$=u_n(a-b+x).$$

Also $\quad v_1-a=\dfrac{a(a+x)}{b}-a=\dfrac{a(a-b+x)}{b}=u_1(a-b+x).$

Therefore $u_1+u_2+\ldots+u_n$

$$=\frac{(v_1-a)+(v_2-v_1)+\ldots+(v_n-v_{n-1})}{a-b+x}=\frac{v_n-a}{a-b+x}$$

$$=\{u_n(a+nx)-a\}/(a-b+x).$$

Examples on Chapter X

Sum the following series to n terms

1. $1.3+3.5+5.7+\ldots.$
2. $2.5.8+5.8.11+8.11.14+\ldots.$
3. $1.5.9+5.9.13+9.13.17+\ldots.$
4. $1.3.5.7+3.5.7.9+5.7.9.11+\ldots.$ [*Annam.*,'53]
5. $2.4.6^2+4.6.8^2+6.8.10^2+\ldots.$
6. $1.4.7+2.5.8+3.6.9+\ldots.$
7. $1.2.3+4.5.6+7.8.9+\ldots.$

Sum the following series to n terms and to infinity.

8. $\dfrac{1}{1.3}+\dfrac{1}{3.5}+\dfrac{1}{5.7}+\ldots.$
9. $\dfrac{1}{2.4}+\dfrac{1}{3.5}+\dfrac{1}{4.6}+\ldots.$ [*Andhra*, 1955]
10. $\dfrac{1}{1.3.5}+\dfrac{1}{3.5.7}+\dfrac{1}{5.7.9}+\ldots.$ [*Karnatak*, 1955]

11. $\dfrac{1}{5\,.\,9\,.\,13}+\dfrac{1}{9\,.\,13\,.\,17}+\dfrac{1}{13\,.\,17\,.\,21}+\ldots.$ [*Andhra*,'55]

12. $\dfrac{1}{1\,.\,3\,.\,5\,.\,7}+\dfrac{1}{3\,.\,5\,.\,7\,.\,9}+\dfrac{1}{5\,.\,7\,.\,9\,.\,11}+\ldots.$

[*Annam.*, 1952]

13. $\dfrac{1}{1\,.\,5}+\dfrac{1}{3\,.\,7}+\dfrac{1}{5\,.\,9}+\ldots.$

14. $\dfrac{3}{1\,.\,2\,.\,4}+\dfrac{4}{2\,.\,3\,.\,5}+\dfrac{5}{3\,.\,4\,.\,6}+\ldots.$

15. Sum to infinity the series

$\dfrac{5}{1\,.\,2\,.\,3}+\dfrac{7}{3\,.\,4\,.\,5}+\dfrac{9}{5\,.\,6\,.\,7}+\ldots.$ [*Mysore*, 1952]

16. If $a\neq 2$, sum to n terms the series

$\dfrac{1}{a}+\dfrac{1\,.\,2}{a(a+1)}+\dfrac{1\,.\,2\,.\,3}{a(a+1)(a+2)}+\ldots.$ [*Andhra*, '52]

Sum to n terms the series

17. $\dfrac{5}{6}+\dfrac{5\,.\,7}{6\,.\,8}+\dfrac{5\,.\,7\,.\,9}{6\,.\,8\,.\,10}+\ldots.$ [*Annam.*, 1952]

18. $\dfrac{4}{5}+\dfrac{4\,.\,7}{5\,.\,8}+\dfrac{4\,.\,7\,.\,10}{5\,.\,8\,.\,11}+\ldots.$

MISCELLANEOUS EXAMPLES

1. Show that the following series are equal :

$$1+\frac{1}{4}+\frac{1.4}{4.8}+\frac{1.4.7}{4.8.12}+\frac{1.4.7.10}{4.8.12.16}+\dots,$$

and

$$1+\frac{2}{6}+\frac{2.5}{6.12}+\frac{2.5.8}{6.12.18}+\frac{2.5.8.11}{6.12.18.24}+\dots. \quad \text{[\textit{Alld.}, '53]}$$

2. Find the sum of the series

$$\frac{1}{6}+\frac{1.4}{6.12}+\frac{1.4.7}{6.12.18}+\dots. \qquad \text{[\textit{Karnatak}, 1959]}$$

3. If $c_0, c_1, c_2, \dots, c_n$ are the coefficients in the expansion of $(1+x)^n$ when n is a positive integer, prove that

(i) $c_0-2c_1+3c_2-4c_3+\dots+(-1)^n(n+1)c_n=0$. [*Utkal*,'52]

(ii) $c_0c_r+c_1c_{r+1}+c_2c_{r+2}+\dots+c_{n-r}c_n=(2n)!/(n-r)!(n+r)!$

4. Show that

$$\frac{1}{1!(n-1)!}+\frac{1}{2!(n-2)!}+\frac{1}{3!(n-3)!}+\dots+\frac{1}{(n-1)!1!}=\frac{2^n-2}{n!}.$$

5. If $x<1$, expand $\{\log_e(1+x)\}^2$ in ascending powers of x.

[*Agra*, 1953]

6. Show that

$$\log_e\sqrt{12}=1+(\tfrac{1}{2}+\tfrac{1}{3})\tfrac{1}{4}+(\tfrac{1}{4}+\tfrac{1}{5})\frac{1}{4^2}+(\tfrac{1}{6}+\tfrac{1}{7})\frac{1}{4^3}+(\tfrac{1}{8}+\tfrac{1}{9})\frac{1}{4^4}+\dots.$$

[*Rajputana*, 1953]

7. If s_n represents the sum of the products of the first n natural numbers taken two at a time, prove that

$$\frac{s_2}{3!}+\frac{s_3}{4!}+\frac{s_4}{5!}+\dots+\frac{s_n}{(n+1)!}+\dots=\frac{11}{24}e.$$

[*U.P.C.S.*, 1946]

8. The numbers x, y, z are real and not all equal. Prove that

$$10x^2+5y^2+13z^2>2(xy+4yz+9zx).$$

[*U.P.C.S.*, 1948]

9. If $1-2x$, $1-4y$ and $x+2y$ be positive, prove that $(1-2x)(1-4y)(x+2y)$ cannot be greater than $\frac{4}{27}$.

[*Annamalai*, 1949]

10. Prove that

$$\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdots\frac{2n-1}{2n}<\frac{1}{\sqrt{(2n+1)}}.$$

11. If a, b, c are positive, prove that

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\geqslant\frac{3}{2}.$$

[*Sagar*, 1951]

12. Show that

$$\frac{a^8+b^8+c^8}{a^3b^3c^3}>\frac{1}{a}+\frac{1}{b}+\frac{1}{c},$$

where a, b, c are positive numbers. [*Agra*, 1952]

13. If $0<x<y$, prove that

$$\frac{\log(1+y)}{\log y}<\frac{\log(1+x)}{\log x}.$$

14. Show that

$$\left(\frac{a+b+c+\ldots+k}{n}\right)^{a+b+c+\ldots+k}<a^ab^bc^c\ldots k^k.$$

[*Nagpur*, '54]

15. If s be the sum of n positive unequal numbers a, b, c, ... then show that

$$\frac{s}{s-a}+\frac{s}{s-b}+\frac{s}{s-c}+\ldots>\frac{n^2}{n-1}.$$

[*Banaras*, '58]

16. If a_1, a_2, ..., a_n are positive and $a_1+a_2+\ldots+a_n\leqslant 1$, prove that

$$\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\geqslant n^2.$$

[*Lucknow*, 1953]

17. If a, b, c are any real numbers, show that

$$(b+c-a)^2+(c+a-b)^2+(a+b-c)^2 \geqslant bc+ca+ab.$$

[*Aligarh*, 1960]

18. If a, b, c be positive rational numbers and the sum of any two greater than the third, show that

$$\left(1+\frac{b-c}{a}\right)^a \left(1+\frac{c-a}{b}\right)^b \left(1+\frac{a-b}{c}\right)^c < 1. \quad \text{[I.A.S., 1960]}$$

19. Prove that

$$a^7+b^7+c^7 > abc(a^4+b^4+c^4),$$

where a, b, c are positive quantities. [*Lucknow*, 1960]

20. Resolve into partial fractions

$$\frac{x^3}{(x+2)^2(x^2+2)}. \quad \text{[Banaras, 1952]}$$

21. Resolve into partial fractions

$$\frac{9x^3-24x^2+48x}{(x-2)^4(x+1)}. \quad \text{[Agra, 1958]}$$

22. Resolve into partial fractions

$$\frac{x}{(x-1)(x^2+1)^2}. \quad \text{[Rajasthan, 1960]}$$

23. Resolve $x^3/(x-a)(x-b)(x-c)$ into partial fractions. Hence show that

$$\frac{a^3}{(a-b)(a-c)(a-d)}+\frac{b^3}{(b-a)(b-c)(b-d)}+\frac{c^3}{(c-a)(c-b)(c-d)}$$
$$+\frac{d^3}{(d-a)(d-b)(d-c)}=1. \quad \text{[Banaras, 1953]}$$

24. If $$\frac{1}{(1-ax)(1-bx)}=\frac{A}{1-ax}+\frac{B}{1-bx},$$

prove that $$\frac{1}{(1-ax)^2(1-bx)}=\frac{A}{(1-ax)^2}+\frac{AB}{1-ax}+\frac{B^2}{1-bx}.$$

[*Madras*, 1951]

25. Show that if $|x|<1$, the coefficient of x^{2n} in the expansion in ascending powers of x, of $x^2/(x^2+1)(x-1)^3$ is

$$\tfrac{1}{4}\{1-2n-4n^2+(-1)^{n-1}\}. \quad \text{[U.P.C.S., 1959]}$$

26. Expressing $\dfrac{x^2+x+1}{x^2+1}$

as a simple continued fraction, find two polynomials A and B such that

$$A(x^2+x+1)-B(x^2+1)=-1. \qquad [U.P.F.S., 1959]$$

27. Find the value of

$$\left(\frac{1}{a+}\,\frac{1}{b+}\,\frac{1}{c+}\,\frac{1}{a+}\cdots\right)\left(c+\frac{1}{b+}\,\frac{1}{a+}\,\frac{1}{c+}\cdots\right). \qquad [I.A.S., 1960]$$

28. Find the nth convergent to

$$\frac{1}{3+}\,\frac{1}{3+}\,\frac{1}{3+}\,\frac{1}{3+}\cdots. \qquad [Allahabad, 1953]$$

29. Express 763/396 as a continued fraction, and hence find the least positive integral values of x and y which satisfy

$$396x-763y=12. \qquad [Agra, 1952]$$

Test the convergence of the following series:

30. $\displaystyle\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^n.$ [*Lucknow*, 1952]

31. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{(1+1/n)^n}.$ [*Madras*, 1960]

32. $\displaystyle\sum\sqrt{\left(\frac{2^n-1}{3^n-1}\right)}.$ [*Aligarh*, 1960]

33. $2-\frac{3}{2}+\frac{4}{3}-\frac{5}{4}+\frac{6}{5}-\cdots.$ [*Rajasthan*, 1959]

34. $\dfrac{x}{a\,.\,1^2+b}+\dfrac{2x^2}{a\,.\,2^2+b}+\cdots+\dfrac{nx^n}{a\,.\,n^2+b}+\cdots.$

[*Banaras*, 1958]

35. $\dfrac{a}{b}+\dfrac{a(a+d)}{b(b+d)}x+\dfrac{a(a+d)(a+2d)}{b(b+d)(b+2d)}x^2+\cdots.$

[*Allahabad*, 1959]

36. $\frac{x}{2}+\frac{1.3}{2.4}\frac{x^3}{6}+\frac{1.3.5.7}{2.4.6.8}\frac{x^5}{10}+\frac{1.3.5.7.9.11}{2.4.6.8.10.12}\frac{x^7}{14}+\ldots$ [*I.A.S.*, 1960]

37. Test the convergence of the series

$$x+x^{1+1/2}+x^{1+1/2+1/3}+\ldots.$$ [*Rajputana*, 1954]

38. If Σu_n is a convergent series of positive terms, prove that Σu_n^2 will also be convergent.

39. Prove that

$$\begin{vmatrix} 0 & \alpha & \beta & \gamma \\ l & 0 & c & -b \\ m & -c & 0 & a \\ n & b & -a & 0 \end{vmatrix} = -(a\alpha+b\beta+c\gamma)(al+bm+cn).$$ [*Allahabad*, 1955]

40. Resolve into factors the determinant

$$\begin{vmatrix} a^3 & 3a^2 & 3a & 1 \\ a^2 & a^2+2a & 2a+1 & 1 \\ a & 2a+1 & a+2 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix}.$$ [*Patna*, 1953]

41. Show that

$$\begin{vmatrix} x & l & m & 1 \\ \alpha & x & n & 1 \\ \alpha & \beta & x & 1 \\ \alpha & \beta & \gamma & 1 \end{vmatrix} = (x-\alpha)(x-\beta)(x-\gamma)$$

where l, m, n have any values whatever. [*Aligarh*, 1960]

42. If $ax+by+cz=1$, $bx+cy+az=0$, $cx+ay+bz=0$, prove that

$$\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix} . \begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a \end{vmatrix} = 1.$$ [*Gujarat*, 1959]

43. Prove that

$$\begin{vmatrix} a^2-bc & b^2-ca & c^2-ab \\ c^2-ab & a^2-bc & b^2-ca \\ b^2-ca & c^2-ab & a^2-bc \end{vmatrix} = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}^2.$$ [*Sagar*, 1951]

44. Show that

$$\begin{vmatrix} 0 & (\alpha-\beta)^2 & (\alpha-\gamma)^2 \\ (\beta-\alpha)^2 & 0 & (\beta-\gamma)^2 \\ (\gamma-\alpha)^2 & (\gamma-\beta)^2 & 0 \end{vmatrix} = 2[(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)]^2.$$

[*Rajasthan*, 1960]

45. Express

$$\begin{vmatrix} b^2+c^2 & ab & ac \\ ba & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix}$$

as the square of a determinant and hence evaluate it.

[*Poona*, 1960]

46. Solve by the method of determinants, the equations

$$\begin{aligned} 2x+3y-4z+16&=0, \\ 3x-y+2z-11&=0, \\ x+y+z-2&=0. \end{aligned}$$

[*Karnatak*, 1959]

47. Solve completely the equations

$$\begin{aligned} ax+by+cz&=k, \\ a^2x+b^2y+c^2z&=k^2, \\ a^3x+b^3y+c^3z&=k^3, \end{aligned}$$

by making use of determinants. [*Kashmir*, 1952]

48. Prove that the product of the two matrices

$$\begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}, \begin{bmatrix} \cos^2\phi & \cos\phi\sin\phi \\ \cos\phi\sin\phi & \sin^2\phi \end{bmatrix}$$

is zero when θ and ϕ differ by an odd multiple of $\frac{1}{2}\pi$.

[*Rajasthan*, 1960]

49. For the matrices

$$A=\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix} 2 & 5 & 3 \\ 3 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

show that $(AB)^{-1}=B^{-1}A^{-1}$. [*Calcutta*, 1960]

50. Sum to n terms the series

$$\frac{4}{1.2}\left(\frac{2}{3}\right)+\frac{5}{2.3}\left(\frac{2}{3}\right)^2+\frac{6}{3.4}\left(\frac{2}{3}\right)^3+\ldots$$

[*Gauhati*, 1960]

THEORY OF EQUATIONS

CHAPTER XI

PROPERTIES OF EQUATIONS & ROOTS

11·1. Preliminary. A function of x put equal to zero is generally called an *equation* if the function is equal to zero only for special values of x. Thus $x^2-3x+2=0$ is an equation, because x^2-3x+2 is actually equal to zero only when x is equal to 2 or 1. For other values of x this expression is not equal to zero.

Since some terms of an equation may be transposed to the right-hand side, an equation may be written in other forms also. For example, $x^2+2=3x$ is also an equation; but it is convenient to transpose all terms to the left, and let the right-hand side be zero.

If $f(x)$ is equal to zero for every value of x, then $f(x)=0$ is called an *identity*. For example

$$x^2-3x+2-(x-1)(x-2)=0$$

is an identity. Often identities also are referred to as equations, but in the strict sense this is wrong. If it is necessary to indicate that a relation is an identity, the symbol $\equiv$ is used. Thus $f(x)\equiv 0$ means that $f(x)$ is identically zero.

If n is a positive integer, the expression

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n,$$

where $a_0, a_1, \ldots, a_n$ are constants, is called a *polynomial*. Some of the constants $a_1, a_2, \ldots$ may be zero, though a_0 is always taken as non-zero. When the polynomial contains only two terms it is termed as a *binomial*.

A polynomial equated to zero gives us a *rational integral algebraic equation*.

Here the word rational denotes that fractional powers of x (like $\sqrt{x}$, $x^{2/3}$, etc.) do not occur in the equation. The word integral denotes that fractions (like $(ax+b)/(cx+d)$ or more complicated ones) do not occur. The word algebraic denotes that trigonometric functions like $\sin x$, $\cos x$, etc., or logarithmic or exponential functions (like a^x) do not occur. Only rational integral algebraic equations will be considered in this and subsequent chapters.

Any value of x which satisfies an equation is called a *root* of that equation. The determination of all the roots of an equation constitutes, the *solution* of the equation.

We shall take the standard form of our algebraic equation to be

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n=0,$$

or, after division by a_0,

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n=0.$$

From now onwards we shall use $f(x)$ to denote the left-hand side of the last equation. The term p_n in $f(x)$, which does not contain x, is called the *absolute term*.

The degree n of the highest term in an equation is called the *degree of the equation*.

Thus $x^2-3x+2=0$ is an equation of the second degree, and $x^3-1=0$ is an equation of the third degree. Equations of degree 2, 3 and 4 respectively are also called *quadratic*, *cubic* and *biquadratic* equations.

11·2. Division of a Polynominal by a Binomial. It is often required to divide a polynomial with numerical coefficients by a binomial with

numerical coefficients. It is important, therefore, to have a method by which this could be carried out quickly and conveniently. To obtain such a method, suppose that when

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n \tag{1}$$

is divided by $x-h$, the quotient is

$$b_0x^{n-1}+b_1x^{n-2}+\ldots+b_{n-2}x+b_{n-1}, \tag{2}$$

and the remainder is R. Then

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots+a_{n-1}x+a_n$$
$$\equiv(x-h)(b_0x^{n-1}+b_1x^{n-2}+\ldots+b_{n-2}x+b_{n-1})+R.$$

Equating the coefficients of the various powers of x, we get

$$\left.\begin{array}{ll} b_0=a_0, & b_1=a_1+b_0h, \\ b_2=a_2+b_1h, & \ldots \\ b_{n-1}=a_{n-1}+b_{n-2}h, & R=a_n+b_{n-1}h. \end{array}\right\} \tag{3}$$

These relations will enable us to find b_0, b_1, b_2, ..., and R. The working can be conveniently arranged as follows :

$$\begin{array}{cccccc} a_0 & a_1 & a_2 & \ldots & a_{n-1} & a_n \\ & b_0h & b_1h & & b_{n-2}h & b_{n-1}h \\ \hline b_0 & b_1 & b_2 & \ldots & b_{n-1} & R \end{array}.$$

In the first line are written the successive coefficients of the expression (1). Now a_0 (or b_0, which is equal to it) is multiplied by h, and the product is written below a_1. The sum of these two (a_1+b_0h) gives b_1 by (3). Next b_1 is multiplied by h and the product written below a_2. Their sum gives b_2. A repetition of this process gives all the coefficients of the quotient (2), the last sum thus obtained being the remainder R.

The student should divide $a_0x^n+a_1x^{n-1}+\ldots+a^n$ by $x-h$ by the method of long division and compare his work with the above. He will notice then that the above is only a convenient way of writing the various steps in one line.

This method is useful in certain important operations in the Theory of Equations (see §§ 11·21, 12·14, 14·5).

Ex. Find the quotient and the remainder when $3x^4+11x^3+23x^2+21x-10$ is divided by $x+3$.

Here $h=-3$; and the calculation may be arranged as follows :

$$\begin{array}{ccccc} 3 & 11 & 23 & 21 & -10 \\ & -9 & -6 & -51 & 90 \\ \hline & 2 & 17 & -30 & 80 \end{array}$$

Hence the quotient is $3x^3+2x^2+17x-30$ and the remainder is 80.

NOTE. The missing powers, if any, in the polynomial to be divided must be indicated by zero coefficients.

11·21. Evaluation of a Polynomial. Let $f(x)$ be a polynomial which on division by $x-h$ gives a quotient $Q(x)$ and a remainder R. Then R, being of a lower degree than $x-h$, must be a constant. Thus

$$f(x)=(x-h)Q(x)+R.$$

Putting $x=h$ in this relation, we get

$$f(h)=R,$$

i.e., *the value of the polynomial $f(x)$ for $x=h$, is equal to the remainder obtained on dividing $f(x)$ by $x-h$.*

This is quite a convenient way of evaluating a polynomial for specific numerical values of x.

Ex. Find the value of $3x^4+11x^3+23x^2+21x-10$ for $x=-3$.

On dividing the given polynomial by $x+3$ (see ex. above), we obtain the remainder 80. Hence the required value is 80.

11·3. Properties of Equations. (i) *If $f(x)$ is a polynomial, a and b are real, and $f(a)$ and $f(b)$ have opposite signs, one plus and the other minus, then the equation $f(x)=0$ has at least one real root between a and b.*

We know that the polynomial $f(x)$ is a continuous function of x (see Text-Book on Differential Calculus by Gorakh Prasad, Chap. I). So, as x passes through all the values from a to b, $f(x)$ must pass through all the values from $f(a)$ to $f(b)$. But of $f(a)$ and $f(b)$ one is positive and the other negative. So, at least for one value, say c, of x, lying between a and b, $f(x)$ must become 0. Then c is the required root.

The student should draw a graph. If he would attempt to join the point $\{a, f(a)\}$ and $\{b, f(b)\}$ by any continuous curve, simple or complicated, in passing from $f(a)$ to $f(b)$ his curve is bound to cut the x-axis ($y=0$, i.e., $f(x)=0$) at least once. It may cut it 3 or 5 or any odd number of times, and so between a and b there must be at least one root of $f(x)=0$, but there may be any odd number of roots.

(ii) *Every equation of an odd degree has at least one real root of a sign opposite to that of its absolute term.*

Let the equation be $f(x)=0$, where $f(x)\equiv x^n+p_1x^{n-1}+\ldots+p_n$. The absolute term p_n does not contain x or any power of x as a factor. Hence $f(0)=p_n$.

Also $f(+\infty)=+\infty$, because when x is very large, x^n is much larger than $p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n$, and therefore $x^n+p_1x^{n-1}+\ldots+p_n$ is large and positive even if $p_1x^{n-1}+\ldots+p_n$ is negative.

Similarly $f(-\infty)=-\infty$.

CASE I. If p_n is positive, we see by (i) above that there is at least one root between $x=-\infty$ and

$x=0$, because $f(-\infty)$ is negative and $f(0)$ is positive, i.e., there is in this case at least one negative root.

CASE II. If p_n is negative we see that now $f(0)$ is negative and $f(\infty)$ is positive, so that now there is at least one positive root, which proves the proposition.

NOTE. Strictly speaking ∞ is not a number: the student must note that we have used ∞ in the above only for "a sufficiently large number." We shall do so in the future also without pointing this out.

(iii) *Every equation of an even degree, whose absolute term is negative, has at least two real roots, one positive and one negative.* For now $f(-\infty)$ is positive, $f(0)$ is negative, and $f(+\infty)$ is positive, on account of x^n being positive both for $x=+\infty$ and $x=-\infty$. It follows by (i) that at least one root lies between $x=-\infty$ and $x=0$, and at least one root lies between $x=0$ and $x=+\infty$.

11·31. Number of Roots. (i) We shall assume here that *every algebraic equation has at least one root*, although that root may be a complex number of the form $a+ib$. The proof of this fundamental theorem is difficult and beyond the scope of the present volume.

(ii) *Every equation $f(x)=0$ of degree n has exactly n roots.* Let a_1 be a root (by (i) this must exist). Divide $f(x)$ by $x-a_1$. Then if $Q_1(x)$ is the quotient and R the remainder, we know that R must be of a lower degree than $x-a_1$; i.e. it cannot involve x; it must be a constant. Thus

$$f(x)\equiv(x-a_1)Q_1(x)+R.$$

Putting $x=\alpha_1$ in this, we find that $f(\alpha_1)=R$. But by hypothesis α_1 is a root of $f(x)=0$. So $f(\alpha_1)=0$. Therefore also $R=0$. Hence

$$f(x)\equiv(x-\alpha_1)Q_1(x).$$

The equation $f(x)=0$ now reduces to

$$(x-\alpha_1)Q_1(x)=0,$$

which is equivalent to $x-\alpha_1=0$ and $Q_1(x)=0$. Then by (i) above, $Q_1(x)=0$ must have a root, say α_2. Proceeding as before, we can show that $f(x)$ is equivalent to

$$(x-\alpha_1)(x-\alpha_2)Q_2(x)=0;$$

and so on. We shall finally find that $f(x)=0$ is equivalent to

$$(x-\alpha_1)(x-\alpha_2)\dots(x-\alpha_n)=0,$$

which has exactly n roots. So the equation $f(x)=0$ also has exactly n roots.

Some of these roots may be complex (imaginary), and some may be equal to one another.

COROLLARIES (i) *If the roots of $f(x)=0$ are $\alpha, \beta, \gamma, \dots$, then*

$$f(x)\equiv(x-\alpha)(x-\beta)(x-\gamma)\dots .$$

(ii) If a polynomial $f(x)$ of degree n vanishes for more than n different values of x, it must be identically zero, for none of the factors $x-\alpha$, $x-\beta$, ... can vanish by substituting for x a value different from $\alpha, \beta, \dots$.

(iii) If two polynomials in x, each of the nth degree, be equal to each other for more than n values of x, the polynomials must be identically equal, i.e., the coefficient of every power of x in one must be equal to the corresponding coefficient in the other.

11·32. Imaginary roots. *If the coefficients in $f(x)$ are all real, and $\alpha+i\beta$ is one of the roots of $f(x)=0$, then $\alpha-i\beta$ must also be a root*, i.e., imaginary roots occur in pairs. Here i stands for $\sqrt{(-1)}$.

Multiplication shows that

$$\{x-(\alpha+i\beta)\}\{x-(\alpha-i\beta)\}=(x-\alpha)^2+\beta^2.$$

Now divide $f(x)$ by $(x-\alpha)^2+\beta^2$. Let the quotient be $Q(x)$ and the remainder, which cannot be of a degree higher than 1, be R_1x+R_2, R_1 and R_2 being real numbers.

Then $$f(x)\equiv\{(x-\alpha)^2+\beta^2\}\,Q(x)+R_1x+R_2.$$

Put in this $x=\alpha+i\beta$. Since, by hypothesis, $\alpha+i\beta$ is a root, $f(\alpha+i\beta)=0$. Also this value of x makes $(x-\alpha)^2+\beta^2$ equal to zero, since $x-(\alpha+i\beta)$ is one of its factors. We get therefore

$$0\equiv R_1(\alpha+i\beta)+R_2.$$

Equating the real and imaginary parts of the right-hand side separately to zero, and solving, we see that

$$R_1=0 \quad\text{and}\quad R_2=0.$$

This shows that $(x-\alpha)^2+\beta^2$ is a factor of $f(x)$, i.e., $x-(\alpha+i\beta)$ and $x-(\alpha-i\beta)$ both are factors of $f(x)$ showing that $\alpha+i\beta$ and $\alpha-i\beta$ both are roots of $f(x)=0$.

Corollary. A similar proof shows that if the coefficients of $f(x)=0$ are rational, surd roots of the form $\gamma\pm\sqrt{\delta}$ occur in pairs.

Ex. Solve $3x^3-4x^2+x+88=0$, one root being

$2+\sqrt{(-7)}$ [*Gujarat*, 1959]

Since the given equation has real coefficients, $2-\sqrt{(-7)}$ is also a root of the given equation. Hence two of the factors of the left-hand side of the given equation are

$$(x-2-i\sqrt{7}) \quad\text{and}\quad (x-2+i\sqrt{7}).$$

Their product, i.e. $(x-2)^2+7$,

or $$x^2-4x+11$$

must, therefore, be a factor.

Dividing out the given equation by $x^2-4x+11$, we get $3x+8=0$. Hence the remaining root is $-8/3$. Thus, the roots are

$$2\pm i\sqrt{7},\ -\tfrac{8}{3}.$$

11·4. Descartes' Rule of Signs. *The number of the positive roots of the equation $f(x)=0$ cannot exceed the number of changes of sign (from + to − or from − to +) in the terms occurring in $f(x)$.**

To illustrate what is meant by the number of changes of sign, consider the following equations :

$$x^3-2x^2+3x-7=0,$$
$$x^3+2x^2+3x-7=0,$$
$$x^3-2x^2+3x+7=0.$$

In the first equation the first term has the sign +, the next −, the next after that + and the last one −. Writing these signs consecutively, we have + − + −. Counting the changes of sign, we see that there are 3 changes of sign. Similarly, the second equation has only one change of sign, and the last one two changes of sign.

Consider the case of any equation taken at random. Suppose the signs of the terms are as follows :

$$+ + + - + - - - - + + - - .$$

Multiply the equation by $x-a$ where a is any positive number. The signs of the terms in the multiplication will be as shown in the following scheme.

$$\begin{array}{cccccccccccccc}
+ & + & + & - & + & - & - & - & - & + & + & - & - & \\
+ & - & & & & & & & & & & & & \\
\hline
+ & + & + & - & + & - & - & - & - & + & + & - & - & \\
 & - & - & - & + & - & + & + & + & + & - & - & + & + \\
\hline
+ & \pm & \pm & - & + & - & \mp & \mp & \mp & + & \pm & - & \mp & +
\end{array} \qquad (1)$$

*This rule is called Descartes' Rule of Signs after Rene Descartes (1596-1650) the great French mathematician and philosopher after whom the Cartesian axes are named.

In the sum, certain terms are certainly positive, being the sum of two positive terms; others are similarly certainly negative; but in the case of those terms which are obtained by adding two terms of different signs (one positive and the other negative), we cannot say for certain whether the sum is positive or negative: the sign will depend upon the numerical values of the terms. The sign has, therefore, been indicated by $\pm$ or $\mp$, showing that it may be positive or negative. Also, if the given equation contains n signs, the sum (1) will contain $n+1$ signs.

Now if we deliberately assign to the ambiguous signs $\pm$ such signs ($+$ or $-$) as will give the least number of changes of sign in (1), there cannot be less changes of sign between the first n signs in (1) than the number of changes of sign in the given equation. [The student will get convinced of this at once when he tries to minimise the number of changes. These will be obtained by taking the upper sign in $\pm$ or $\mp$ everywhere in (1).] Moreover, there is an extra sign in (1) and that certainly involves one more change of sign.

Thus multiplication of $f(x)$ by $x-\alpha$, where α is positive, has increased the number of changes of sign by at least one.

Suppose now that a polynomial is formed by multiplying out the factors which give negative and imaginary roots. Then, if $\alpha, \beta, \gamma, \ldots$ are the positive roots, successive multiplications by $x-\alpha$, $x-\beta$, ... will each increase the number of changes of sign by at least one. Hence in the complete equation there will be at least as many changes of sign as it has positive roots.

COROLLARY. *The number of negative roots of $f(x)=0$ cannot exceed the number of changes of sign in $f(-x)$.*

For, if $f(x)\equiv(x-\alpha)(x-\beta)(x-\gamma)\ldots$, then

$$f(-x)=(-1)^n(x+\alpha)(x+\beta)(x+\gamma)\ldots.$$

Therefore the roots of $f(-x)=0$ are $-\alpha$, $-\beta$, ..., i.e., are numerically equal to the roots of $f(x)=0$, but of the opposite sign. Consequently the number of negative roots of $f(x)=0$ is the same as the number of positive roots of $f(-x)=0$.

If the number of positive and negative roots of an equation of degree n is found by Descartes' rule to be not more than n', where $n'<n$, we can infer at once that at least $n-n'$ of the roots of $f(x)=0$ are imaginary.

Ex. Show that the equation $x^4-2x^3-1=0$ has at least two imaginary roots.

As there is only one change of sign in x^4-2x^3-1, the equation under consideration cannot have more than one positive root.

Also, writing $-x$ for x, we get x^4+2x^3-1, which also has only one change of sign. So the given equation cannot have more than one negative root.

As the given equation is of the fourth degree, it must have four roots. It follows that at least two of the roots are imaginary.

EXAMPLES

1. Find the quotient and the remainder when $3x^4-5x^3+10x^2+11x-61$ is divided by $x-3$.

2. Find the quotient and the remainder when x^5-2x^4+3 is divided by $x+2$.

[Hint. The coefficients to be written in the first line in this case are 1, -2, 0, 0, 0, 3.]

3. If $f(x)=x^3+5x^2+3x-2$, find $f(-1)$ and $f(-2)$.

4. Evaluate $3x^4+2x^3-6x-4$ for $x=1{\cdot}2$.

5. Show that $x^3-7x+2=0$ has one negative root, a positive root between 0 and 1, and another positive root greater than 1.

6. Form the equation whose roots are -3, -1, $\frac{5}{3}$.

7. Form an equation with rational coefficients two of whose roots are $1+\sqrt{2}$ and $-1-3i$.

8. Solve the equation

$$x^4-16x^3+86x^2-176x+105=0,$$

two roots being 1 and 7.

9. Solve $2x^4-4x^3+11x^2-9x-26=0$, one root being $\frac{1}{2}+\frac{5}{2}i$.

10. Solve $6x^4-13x^3-35x^2-x+3=0$. one root being $2-\sqrt{3}$. [*Kashmir*, 1953]

11. Show that the equation $x^5+x^3-2x^2+x-2=0$ has at least one pair of imaginary roots. [*Calcutta*, 1960]

12. Show that $2x^7-x^4+4x^3-5=0$ has at least four imaginary roots. [*Gorakhpur*, 1960]

13. Show that $x^6-x^5-10x+7=0$ has two positive and four imaginary roots. [*Allahabad*, 1959]

11·5. Relations between Roots and Coefficients. Let the roots of the equation $f(x)=0$, where

$$f(x)\equiv x^n+p_1x^{n-1}+\ldots+p_n,$$

be $\alpha_1, \alpha_2, \ldots, \alpha_n$. Then

$$x^n+p_1x^{n-1}+\ldots+p_n\equiv(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n).$$

Multiplying out the factors on the right-hand side, and equating the coefficients of the various powers of x on the two sides, we get

$$p_1=-(\alpha_1+\alpha_2+\alpha_3+\ldots+\alpha_n),$$
$$p_2=+(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3+\ldots+\alpha_{n-1}\alpha_n),$$

$$p_3=-(a_1a_2a_3+a_1a_3a_4+\dots+a_{n-2}a_{n-1}a_n),$$

$$\dots \qquad \dots \qquad \dots$$

$$p_n=(-1)^n a_1a_2a_3\dots a_n.$$

Denoting $a_1+a_2+\dots+a_n$ by Σa_1, $a_1a_2+a_1a_3+\dots+a_{n-1}a_n$ by Σa_1a_2, and so on, the above result can be written briefly as

$$p_1=-\Sigma a_1,\ p_2=\Sigma a_1a_2,\ p_3=-\Sigma a_1a_2a_3,\ \dots,$$
$$p_n=(-1)^n a_1a_2a_3\dots a_n.$$

NOTE. (i) $\Sigma a_1a_2\dots a_r$ means the sum of all terms of the type $a_1a_2\dots a_r$.

In every equation IN WHICH THE COEFFICIENT OF THE HIGHEST DEGREE TERM IS UNITY, *the coefficient of the second term with its sign changed is equal to the sum of the roots; the coefficient of the third term is equal to the sum of the products of the roots taken two at a time; and so on.*

(ii) The above results are true only when the coefficient of x^n is unity. If this coefficient is different from unity, care must be taken to divide out the equation by that coefficient before applying the formulae.

11·51. Application to the solution of an equation. The student may think that the n relations obtained above would be of help in obtaining all the roots and thus solving the equation. However, this is not so, unless some additional relation or relations between the roots are given.

For example, let α, β, γ be the roots of the equation

$$x^3+p_1x^2+p_2x+p_3=0. \tag{1}$$

Then we have

$$p_1=-(\alpha+\beta+\gamma),$$
$$p_2=\alpha\beta+\alpha\gamma+\beta\gamma,$$
$$p_3=-\alpha\beta\gamma.$$

If we wish to find α from these, we may substitute for $\beta+\gamma$ and $\beta\gamma$ from the first and third equations in the second and get

$$p_2=-\alpha(p_1+\alpha)-p_3/\alpha,$$

or

$$\alpha^3+p_1\alpha^2+p_2\alpha+p_3=0.$$

This equation is the same as equation (1), only α occurs here in place of x, and therefore is equally difficult to solve. A similar result will be obtained on starting from an equation of any other degree.

When an additional relation between the roots is given it can be used in conjunction with the other relations to obtain an equation of a lower degree than the original equation. This is illustrated in the example given below.

Ex. Solve the equation $x^3-3x+2=0$, given that two of its roots are equal.

If the roots are α, α and β, two of the relations between the roots and the coefficients of the equation become

$$2\alpha+\beta=0,$$

and

$$\alpha^2+2\alpha\beta=-3.$$

Eliminating β, we get

$$\alpha^2+2\alpha(-2\alpha)=-3, \text{ i.e. } \alpha^2=1,$$

giving $\alpha=1$ or -1. The latter root is inadmissible as it does not satisfy the original equation $x^3-3x+2=0$. Now $\alpha=1$ gives $\beta=-2$. So the roots are 1, 1, -2.

11·52. Symmetric Functions of the Roots.

Symmetric functions of the roots of an equation are such functions of all the roots as involve each root in the same way, so that the function is not altered when any two of the roots are interchanged.

Thus $\alpha^2+\beta^2+\gamma^2$ and $\alpha^2\beta+\alpha\beta^2+\beta^2\gamma+\beta\gamma^2+\gamma^2\alpha+\gamma\alpha^2$ are symmetric functions of the three roots α, β and γ, while $\alpha^2\beta+\beta^2\gamma+\gamma^2\alpha$ is not a symmetric function. In the last function, if we interchange α and β we get $\beta^2\alpha+\alpha^2\gamma+\gamma^2\beta$ which is not the same as the original function.

The values of symmetric functions of the roots of an equation can be generally easily found, without finding the roots themselves, by the use of the relations obtained in § 11·5.

Ex. 1. For the cubic $x^3+px^2+qx+r=0$ find the value of $\Sigma\alpha^2$.

We have

$$\alpha+\beta+\gamma=-p.$$

Squaring it, we have $\Sigma\alpha^2+2\Sigma\alpha\beta=p^2$.

Therefore $\Sigma\alpha^2=p^2-2\Sigma\alpha\beta=p^2-2q$.

Ex. 2. If α, β, γ be the roots of the cubic equation $x^3+px^2+qx+r=0$, find the value of

$$\frac{\beta^2+\gamma^2}{\beta+\gamma}+\frac{\gamma^2+\alpha^2}{\gamma+\alpha}+\frac{\alpha^2+\beta^2}{\alpha+\beta}.$$

The given expression

$$=\frac{\Sigma(\beta^2+\gamma^2)(\gamma+\alpha)(\alpha+\beta)}{(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)}=\frac{\Sigma(\beta^2+\gamma^2)(\alpha^2+\alpha\beta+\beta\gamma+\gamma\alpha)}{(\beta+\gamma)(\alpha^2+\alpha\beta+\beta\gamma+\gamma\alpha)}.$$

Now $\Sigma\alpha=-p$ and $\Sigma\alpha\beta=q$; therefore the denominator

$$=(-p-\alpha)(\alpha^2+q)=-(\alpha^3+p\alpha^2+q\alpha+pq)$$
$$=r-pq, \text{ since } \alpha^3+p\alpha^2+q\alpha+r=0.$$

The numerator $=\Sigma(\beta^2+\gamma^2)(\alpha^2+q)=2\Sigma\alpha^2\beta^2+2q\Sigma\alpha^2$

$$=2\{(\Sigma\alpha\beta)^2-2\alpha\beta\gamma\,\Sigma\alpha\}+2q\{(\Sigma\alpha)^2-2\Sigma\alpha\beta\}$$
$$=2\{q^2-2pr\}+2q\{p^2-2q\}=2p^2q-4pr-2q^2.$$

Hence the given expression

$$=\frac{2p^2q-4pr-2q^2}{r-pq}.$$

NOTE. If the student finds any difficulty in seeing why $\Sigma(\beta^2+\gamma^2)(\alpha^2+q)=2\Sigma\alpha^2\beta^2+2q\Sigma\alpha^2$, he should write down the value of the left-hand side in full as

$$(\beta^2+\gamma^2)(\alpha^2+q)+(\gamma^2+\alpha^2)(\beta^2+q)+(\alpha^2+\beta^2)(\gamma^2+q).$$

Multiplying out and collecting similar term see that it is equal to

$$2\Sigma\alpha^2\beta^2+2q\Sigma\alpha^2.$$

He can adopt the same procedure for some of the subsequent steps till he becomes familiar with the Σ notation.

EXAMPLES

1. Solve $x^3-3x^2+4=0$, two of its roots being equal.

2. Solve the equation $x^3-9x^2+23x-15=0$, it being given that the roots are in arithmetical progression. [*Allahabad*, 1960]

3. Solve $x^3-6x^2+3x+10=0$, the roots being in arithmetical progression.

4. Solve $3x^3-26x^2+52x-24=0$, the roots being in geometrical progression. [*Osmania*, 1954]

5. Solve $6x^3-11x^2+6x-1=0$, the roots being in harmonical progression.

6. Solve $x^3-7x^2+36=0$, given that one root is double another.

7. Solve $2x^3+x^2-7x-6=0$, given that the difference of two of the roots is 3. [*Sagar*, 1949]

8. Solve $x^3-5x^2-2x+24=0$, given that the product of two of the roots is 12. [*Delhi*, 1958]

9. The equation $x^4-2x^3+4x^2+6x-21=0$ has two roots equal in magnitude and opposite in sign; determine all the roots. [*Andhra*, 1960]

If α, β, γ be the roots of the equation $x^3+px^2+qx+r=0$, find the value of :

10. $\Sigma 1/\alpha$. [*Mysore*, '48]   11. $\Sigma\alpha^3$. [*Aligarh*, '52]

12. $\Sigma\beta^2\gamma^2$.   13. $\Sigma(\beta^2+\gamma^2)/\beta\gamma$.

14. $(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)$. [*Delhi*, 1958]

15. If α, β, γ, δ be the roots of $x^4+px^3+qx^2+rx+s=0$, find the value of $\Sigma\alpha^2\beta\gamma$.

16. If the roots of $x^4+3px^2+3qx+r=0$ are in harmonical progression, show that $2q^3=r(3pq-r)$. [*Sagar*, 1950]

11·6. Derived Functions. Let $f(x)$ denote the expression

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n. \tag{1}$$

If x is changed to $x+h$, $f(x)$ becomes $f(x+h)$, i.e.,

$$(x+h)^n+p_1(x+h)^{n-1}+p_2(x+h)^{n-2}+\ldots+p_{n-1}(x+h)+p_n,$$

or
$$\begin{aligned} &x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n \\ &+h\{nx^{n-1}+(n-1)p_1x^{n-2}+(n-2)p_2x^{n-3}+\ldots+p_{n-1}\} \\ &+\frac{h^2}{2!}\{n(n-1)x^{n-2}+(n-1)(n-2)p_1x^{n-3}+\ldots+2p_{n-2}\} \\ &+\ldots \end{aligned} \tag{2}$$

It is seen that the terms free from h in the above expression are same as those in (1), i.e. $f(x)$. The coefficient of h in (2) is called the *first derived function of* $f(x)$, and is denoted by $f'(x)$. The coefficient of $h^2/2!$ is called the *second derived function* of $f(x)$, and is denoted by $f''(x)$; and so on. The student will easily see that $f'(x)$ is the first differential coefficient of $f(x)$; $f''(x)$ is the second differential coefficient of $f(x)$; etc., and that the expression (2) is the expansion of $f(x+h)$ by Taylor's theorem.

11·7. Multiple roots. Let the differential coefficient of $f(x)$ be $f'(x)$. Then *a multiple root of order r of the equation $f(x)=0$ is a multiple root of order $r-1$ of the equation $f'(x)=0$.*

If the root α of the equation $f(x)=0$ is a multiple root of order r, i.e., if α occurs r times as a root, then

$$f(x)\equiv(x-\alpha)^r(x-\beta)(x-\gamma)\ldots,$$

β, γ, ... being the other roots of $f(x)=0$.

Differentiation shows that

$$f'(x)=r(x-\alpha)^{r-1}(x-\beta)\ldots+(x-\alpha)^r\frac{d}{dx}\{(x-\beta)\ldots\}.$$

We see that $(x-\alpha)^{r-1}$ is a factor in $f'(x)$. Hence the proposition.

The equation $f'(x)=0$ is called the first *derived equation* of the equation $f(x)=0$; the second, third, etc. derived equations being $f''(x)=0$, $f'''(x)=0$, etc.

Ex. Find the multiple root of

$$x^4-2x^3+2x-1=0,$$

and thus solve the equation.

Here the first derived equation is

$$4x^3-6x^2+2=0, \text{ i.e., } 2x^3-3x^2+1=0.$$

Now the G. C. M. of x^4-2x^3+2x-1 and $2x^3-3x^2+1$ is $(x-1)^2$, as the student can easily verify. Hence $(x-1)^3$ must be a factor of the left-hand side of the given equation. Division shows that the other factor is $x+1$. Hence the roots are 1, 1, 1 and -1.

11·8. Sums of Powers of Roots. If $\alpha, \beta, \gamma, \ldots$ are the roots of the equation $f(x)=0$ we have

$$f(x)\equiv(x-\alpha)(x-\beta)(x-\gamma)\ldots.$$

Taking logarithms of both the sides and differentiating

$$\frac{f'(x)}{f(x)}\equiv\frac{1}{x-\alpha}+\frac{1}{x-\beta}+\frac{1}{x-\gamma}+\ldots. \qquad (1)$$

Now, $(x-\alpha)^{-1}=x^{-1}(1-\alpha x^{-1})^{-1}$

$$=x^{-1}+\alpha x^{-2}+\alpha^2x^{-3}+\ldots+\alpha^n x^{-n-1}+\ldots.$$

Similar expansions can be obtained for the other terms. Substituting in (1) and collecting the coefficients of x^{-n-1}, we get the following rule :

The sum of the nth powers of the roots, i.e., Σa^n, is equal to the coefficient of x^{-n-1} in the expansion of $f'(x)/f(x)$ in powers of x^{-1}.

Ex. Find the sum of the fourth powers of the roots of $x^3-x-1=0$.

Here $f'(x)=3x^2-1$.

Therefore $f'(x)/f(x)=(3x^2-1)/(x^3-x-1)$

$$=(3x^{-1}-x^{-3})(1-x^{-2}-x^{-3})^{-1}$$

$$=(3x^{-1}-x^{-3})\{1+x^{-2}+x^{-3}+(x^{-2}+x^{-3})^2+\ldots\}. \qquad (1)$$

Therefore $\Sigma a^4=$the coefficient of x^{-5} in (1)

$$=3-1=2.$$

Examples

Solve the following equations which have multiple roots:

1. $8x^3+20x^2+14x+3=0$. [*Andhra*, 1960]
2. $x^4-9x^2+4x+12=0$. [*Karnatak*, 1954]
3. $x^4-6x^2+8x-3=0$. [*Mysore*, 1948]
4. $x^4-7x^3+17x^2-17x+6=0$. [*Gorakhpur*, 1959]
5. $8x^4+4x^3-18x^2+11x-2=0$. [*Allahabad*, 1959]
6. $x^5-x^3+4x^2-3x+2=0$.
7. Determine the condition that the cubic
$$z^3+3Hz+G=0$$
should have a pair of equal roots.

8. If the equation $ax^3+3bx^2+3cx+d=0$ has two equal roots, show that each of them is equal to
$$(bc-ad)/2(ac-b^2).$$ [*Gauhati*, 1960]

9. If the equation $x^5-10a^3x^2+b^4x+c^5=0$ has three equal roots, show that $ab^4-9a^5+c^5=0$. [*Nagpur*, 1954]

10. Find the sum of the fourth powers of the roots of
$$x^3-2x^2+x-1=0.$$ [*Sagar*, 1948]

11. Find the sum of the fifth powers of the roots of the equation $x^4-3x^3+5x^2-12x+4=0$. [*Nagpur*, 1949]

12. If α, β, γ be the roots of the cubic $ax^3+cx+d=0$, find the value of $\alpha^5+\beta^5+\gamma^5$. [*Allahabad*, 1960]

***11·9. Sturm's Functions.** Though Descartes' rule of signs gives an upper limit to the number of real roots of an equation, it does not give their exact number, nor does it give their location. Both of these things can be determined by a theorem due to Sturm, which is based on the consideration of the signs of a sequence of functions obtained as follows.

Let $f(x)$ be a polynomial and let $f_1(x)$ be its first derived function. Divide $f(x)$ by $f_1(x)$, and let the quotient be $q_1(x)$ and the remainder $-f_2(x)$. Next divide $f_1(x)$ by $f_2(x)$, and let the quotient be $q_2(x)$ and the remainder $-f_3(x)$. We can next divide $f_2(x)$ by $f_3(x)$ to obtain the quotient $q_2(x)$ and the remainder $-f_4(x)$. This can be repeated till we get a remainder $-f_m(x)$ which is either a constant, or which divides the previous remainder $-f_{m-1}(x)$ completely. Then the functions

$$f(x),\ f_1(x),\ f_2(x),\ f_3(x),\ \ldots,\ f_m(x)$$

are known as *Sturm's functions*.

We see that $f_2(x), f_3(x), \ldots, f_m(x)$ are virtually the successive remainders in the process of finding the G. C. M. of $f(x)$ and $f_1(x)$, except that we change the sign of each remainder before proceeding to the next division, and that $f_2(x)$, $f_3(x)$, ... are the remainders with their signs thus changed. We can write

$$\begin{aligned} f(x) &= q_1(x)f_1(x) - f_2(x), \\ f_1(x) &= q_2(x)f_2(x) - f_3(x), \\ \cdots & \qquad \cdots \\ f_{m-2}(x) &= q_{m-1}(x)f_{m-1}(x) - f_m(x). \end{aligned}$$

Ex. 1. Find the sequence of Sturm's functions for the polynomial

$$f(x) = x^3 - 7x + 7.$$

Here the derived function of $f(x)$ is

$$f_1(x) = 3x^2 - 7.$$

*This and the subsequent articles may be omitted at a first reading.

On division

$$\begin{array}{r}
3x^2-7)\,x^3-7x+7\,(\tfrac{1}{3}x \\
\underline{x^3-\tfrac{7}{3}x} \\
-\tfrac{14}{3}x+7
\end{array}$$

Therefore $f_2(x)=\frac{14}{3}x-7$. Again

$$\begin{array}{r}
\tfrac{14}{3}x-7)\,3x^2-7\,(\tfrac{9}{14}x+\tfrac{27}{28} \\
\underline{3x^2-\tfrac{9}{2}x} \\
\tfrac{9}{2}x-7 \\
\underline{\tfrac{9}{2}x-\tfrac{27}{4}} \\
-\tfrac{1}{4}
\end{array}$$

so that $f_3(x)=\frac{1}{4}$. Hence the required functions are

$$x^3-7x+7,\ 3x^2-7,\ \tfrac{14}{3}x-7,\ \tfrac{1}{4}.$$

NOTE. We shall see in the next article that only the signs of Sturm's functions are required (and not their values). Hence we can always multiply the functions by a *positive* number if convenient. For example, it would be convenient to multiply f_3 and f_4 by $\frac{3}{7}$ and 4 respectively in the above example, and write Sturm's functions as

$$x^3-7x+7,\ 3x^2-7,\ 2x-3,\ 1.$$

Ex. 2. Find Sturm's functions when

$$f(x)=x^4-5x^3+9x^2-7x+2.$$

Here the derived function of $f(x)$ is

$$f_1(x)=4x^3-15x^2+18x-7.$$

To avoid fractions we multiply $f(x)$ by 16 and then divide it by $f_1(x)$. Thus we have

$$\begin{array}{r}
4x^3-15x^2+18x-7)\ 16x^4-80x^3+144x^2-112x+32\,(4x-5 \\
\underline{16x^4-60x^3+\ 72x^2-\ 28x} \\
-20x^3+\ 72x^2-\ 84x+32 \\
\underline{-20x^3+\ 75x^2-\ 90x+35} \\
-\ 3x^2+\ 6x-\ 3
\end{array}$$

Changing the sign and dividing by 3 we may take

$$f_2(x)=x^2-2x+1.$$

Proceeding further, we have

$$
\begin{array}{r}
x^2-2x+1)\,4x^3-15x^2+18x-7\,(4x-7 \\
\underline{4x^3-8x^2+4x} \\
-7x^2+14x-7 \\
\underline{-7x^2+14x-7} \\
\times
\end{array}
$$

Thus $f_2(x)$ is the last remainder, as it divides $f_1(x)$ completely. [In fact it is the G.C.M. of $f(x)$ and $f_1(x)$.] Hence the required functions are

$$x^4-5x^3+9x^2-7x+2,\ 4x^3-15x^2+18x-7,\ x^2-2x+1.$$

11·91. Sturm's Theorem. *If $f(x)$ is a polynomial and a, b are any real numbers ($a<b$), the number of distinct real roots of $f(x)=0$ lying between a and b is equal to the excess of the number of changes of sign in the sequence of Sturm's functions*

$$f(x), f_1(x), f_2(x), \ldots, f_m(x)$$

when a is substituted for x, over the number of changes of sign in the sequence when b is substituted for x.

The proof of this theorem will be given later. We shall take up its application first. The theorem holds for repeated roots also, provided we count each repeated root only once.

Ex. 1. Find the number and location of the real roots of the equation

$$x^3-7x+7=0. \qquad \text{[}\textit{Delhi(Hons.)}\text{, '58]}$$

Here the Sturm's functions are (Ex. 1, p. 203).

$$x^3-7x+7,\ 3x^2-7,\ 2x-3,\ 1.$$

The signs of this sequence for the various values of x are given below.

$$
\begin{array}{ll}
x=-\infty & : \ - \ + \ - \ + \\
x=0 & : \ + \ - \ - \ + \\
x=+\infty & : \ + \ + \ + \ +
\end{array}
$$

There are three changes of sign when $x=-\infty$, two changes of sign when $x=0$, and no change of sign when $x=+\infty$. Hence all the three roots are real : one negative, and two positive.

We have, further,

$$x=-4 : - \quad + \quad - \quad +$$
$$x=-2 : + \quad + \quad - \quad +$$
$$x=2 \quad : + \quad + \quad + \quad +$$

Since one change of sign is lost from $x=-4$ to -2, the negative root lies between these values of x. Similarly, the two positive roots lie between $x=0$ and 2. If necessary these limits can be further closed down.

[note that the sequence of signs for $x=-2$ and $x=0$ are different, but that the number of changes of sign are same in both the cases. So no root lies between -2 and 0.]

Ex. 2. Find the nature of the roots of the equation

$$x^5+2x^4+x^3-x^2-2x-1=0.$$

Here $\qquad f_1=5x^4+8x^3+3x^2-2x-2;$

and it can be easily found by division that

$$f_2=2x^3+7x^2+12x+7,$$
$$f_3=-x^2-6x-5,$$

and $\qquad f_4=-x-1.$

Since f_4 completely divides f_3, $x+1$ is the G.C.M. of f and f_1. Therefore -1 is a double root.

The signs of Sturm's functions are

$$x=-\infty : - \quad + \quad - \quad - \quad +$$
$$x=0 \quad : - \quad - \quad + \quad - \quad -$$
$$x=+\infty : + \quad + \quad + \quad - \quad -$$

The numbers of changes of sign are respectively 3, 2 and 1. There are, therefore, two real distinct roots : one negative and one positive. Hence the equation has a double negative root, a real positive root and two imaginary roots.

Examples

Find the number and location of the real roots of the equation

1. $x^3-2x-5=0.$
2. $2x^3-3x^2-6x-2=0.$

3. $3x^4-6x^2-8x-3=0$.

4. $x^4-2x^3-7x^2+10x+10=0$.

Find the nature of the roots of the equation

5. $x^4-5x^3+9x^2-7x+2=0$.

6. $x^4-3x^2-6x+3=0$.

7. $3x^5+5x^3+2=0$.

11·92. Proof of Sturm's Theorem. We shall first prove the theorem of § 11·91 for the case when $f(x)=0$ has no multiple roots. In this case $f(x)$ and $f_1(x)$ have no common factor. So the last remainder f_m, being their G.C.M., is a constant. Furthermore, by the law of formation of successive remainders, f_m is the G.C.M. of any two consecutive remainders f_r and f_{r+1}. So f_r and f_{r+1} can have no common factor. Hence two consecutive remainders will not vanish for the same value of x.

Suppose that x increases gradually from a to b, and consider the signs of the sequence of functions

$$f(x), f_1(x), f_2(x), \ldots, f_m(x). \tag{1}$$

The signs in the sequence alter only when x passes through a value α for which one (or more) of the functions in (1) vanish. When x passes through such a value the sign of that function alters from $+$ to $-$, or from $-$ to $+$. Three cases arise :

(i) One of the functions $f_r(x)$, other than $f(x)$, vanishes at $x=\alpha$.

(ii) $f(x)$ vanishes at $x=\alpha$.

(iii) Two (or more) of the functions vanish at $x=\alpha$.

Consider the first case where $f_r=0$ for $x=\alpha$. Since $f_{r+1}=q_rf_r-f_{r-1}$, we have at $x=\alpha$

$$f_{r+1}=-f_{r-1},$$

i.e., f_{r+1} and f_{r-1} have opposite signs. Suppose they have the signs $+$ and $-$ respectively. Then, being continuous functions, f_{r+1} and f_{r-1} will have these very signs for values of x slightly less or greater than α.

On the other hand, f_r will have different signs on the two sides of α. Let f_r be +ve for $x<\alpha$ and −ve for $x>\alpha$. Then the sequence f_{r-1}, f_r, f_{r+1} will alter its signs from

$$-++ \text{ to } --+$$

when x, while increasing, passes through the value α. In the sequence $-++$, there is just one change of sign. In the sequence $--+$ also there is one change of sign. Thus the number of changes of sign does not alter when x passes through α.

We have taken above f_{r+1} and f_r to be positive for $x<\alpha$. We could take either, or both, of them negative. The student can verify that in every case we shall get just one change of sign in the sequence f_{r-1}, f_r, f_{r+1} either before the passage through $x=\alpha$, or after it.

Hence the total number of changes of sign in (1) does not alter when x passes through any value of x for which one of the functions $f_1, f_2, \ldots, f_m$ vanishes.

Now consider the case when x passes through a value α for which $f(x)=0$. Then $f(x)$ will have different signs on the two sides of α. Suppose $f(x)$ is +ve for a value of x slightly less than α, and −ve for a value slightly greater than α. In this case $f(x)$ is decreasing as x increases, and $f_1(x)$ being its differential coefficient is negative. Thus $f(x)$, $f_1(x)$ have the signs

$$+- \text{ and } --$$

just before and after x passes through the value α. We see that one change of sign is lost in passing through α, a root of $f(x)=0$.

If instead of being positive, $f(x)$ is −ve for $x<\alpha$, then it is +ve for $x>\alpha$; and its derivative $f_1(x)$ is +ve. The sequence $f(x)$, $f_1(x)$ has then the signs $-+$ and $++$ for $x<\alpha$ and $x>\alpha$ respectively. One change of sign is lost in this case also.

Hence the number of changes of sign in (1) decreases by one each time x passes through a root of $f(x)=0$. This proves the theorem for the case when there are no multiple roots.

When two of the functions, say f_r and f_s, vanish simultaneously for $x=\alpha$, then, as proved earlier, these will not be

consecutive functions. Therefore, we can apply the previous arguments separately to f_r and f_s, and the theorem holds for this case also.

11·93. Sturm's Theorem. Multiple Roots. Suppose $f(x)=0$, has repeated roots, and let

$$f(x)=(x-\alpha)^p(x-\beta)^q(x-\gamma)^r\ldots.$$

Then
$$f_1(x)=p(x-\alpha)^{p-1}(x-\beta)^q(x-\gamma)^r\ldots$$
$$+q(x-\alpha)^p(x-\beta)^{q-1}(x-\gamma)^r+\ldots$$

The G.C.M. of $f(x)$ and $f_1(x)$ is

$$(x-\alpha)^{p-1}(x-\beta)^{q-1}(x-\gamma)^{r-1}\ldots, \qquad (1)$$

and this factor is common to all the functions $f_2, f_3, \ldots, f_m$.

Divide $f, f_1, f_2, \ldots, f_m$ by this common factor and let the quotients be $\phi, \phi_1, \phi_2, \ldots, \phi_m$. Then

(i) $\phi=(x-\alpha)(x-\beta)(x-\gamma)\ldots,$

(ii) $\phi_1=p(x-\beta)(x-\gamma)\ldots+q(x-\alpha)(x-\gamma)\ldots+\ldots, \qquad (2)$

(iii) $\phi_2, \phi_3, \ldots, \phi_m$ obey laws of formation similar to those for $f_2, f_3, \ldots, f_m$, viz.

$$\phi_{r-1}=q_r\phi_r-\phi_{r+1}. \qquad . \quad . \quad . \quad (3)$$

(iv) ϕ_m is a constant, and any two consecutive functions ϕ_r, ϕ_{r+1} do not have a common factor.

Thus, the properties of the functions

$$\phi, \phi_1, \phi_2, \ldots, \phi_m \qquad . \quad . \quad . \quad (4)$$

are the same as those of $f, f_1, f_2, \ldots, f_m$ for the case of non-repeated roots, with one difference that ϕ_1 is not the derivative of ϕ. As far as the number of changes of sign in the sequence $\phi_1, \phi_2, \ldots, \phi_m$ is concerned, it makes no difference at all, because the proof for this part is based on the law of formation (3) which is similar in both the cases.

Also ϕ', the derivative of ϕ, is

$$(x-\beta)(x-\gamma)\ldots+(x-\alpha)(x-\gamma)\ldots+\ldots. \qquad (5)$$

At $x=\alpha$, a root of $\phi=0$, we see from (2) and (5) that

$$\phi_1=p\phi'.$$

Thus ϕ_1 and ϕ' have the same sign at a root of $\phi=0$ (and also in its neighbourhood). Therefore the proof for the second part holds here also, and one change of sign in the sequence (4) is lost when x passes through α.

Furthermore, the number of changes of sign in (4) is the same as that in the sequence

$$f, f_1, f_2, \ldots, f_m; \qquad \ldots \quad (6)$$

because we can get the sequence (6) by multiplying each member of (4) by the *same* factor (1).

Hence the number of changes of sign in (6) lost when x changes from a to b, is equal to the number of roots lying between a and b, *each repeated root being counted only once.*

EXAMPLES ON CHAPTER XI

Find the nature of the roots of the equation

1. $3x^4+12x^2+5x-4=0$. [*Banaras*, 1953]

2. $x^4+15x^2+7x-11=0$. [*Allahabad*, 1957]

3. Use Descartes' rule of signs to show that the equation

$$x^{10}-4x^6+x^4-2x-3=0$$

has at least four unreal roots. [*Nagpur*, 1950]

4. Solve the equation

$$x^4+4x^3+5x^2+2x-2=0$$

of which one root is $-1+\sqrt{(-1)}$. [*Banaras*, 1949]

5. Solve the equation $x^3-3x^2-6x+8=0$, given that the roots are in arithmetical progression. [*Kashmir*, 1954]

6. Solve the equation

$$x^3-13x^2+15x+189=0,$$

given that one root exceeds another by 2. [*U.P.C.S.*, 1946]

7. Solve $x^4-2x^3-21x^2+22x+40=0$, given that the roots are in arithmetical progression.

8. Solve $x^4-2x^3-3x^2+4x-1=0$, given that the product of two of its roots is unity.

9. Solve the equation $x^3+x^2+2x+8=0$, given that its roots are in geometrical progression. [*Mysore*, 1953]

10. Solve the equation

$$3x^3-22x^2+48x-32=0,$$

the roots of which are in harmonic progression. [*Nagpur*, 1954]

11. Solve the equation

$$x^5-3x^4-5x^3+27x^2-32x+12=0,$$

given that there are multiple roots. [*Mysore*, 1949]

12. If the equation $x^4+ax^3+bx^2+cx+d=0$ has three equal roots, show that each of them is equal to

$$(6c-ab)/(3a^2-8b).$$ [*Sagar*, 1949]

13. If the sum of two of the roots of the equation

$$x^4+ax^3+bx^2+cx+d=0$$

equals the sum of the other two, prove that

$$4ab=a^3+8c.$$ [*Bombay*, 1955]

14. Find the condition that the roots of the cubic

$$ax^3+3bx^2+3cx+d=0$$

are in arithmetical progression. [*Gorakhpur*, 1959]

15. If the roots of the equation

$$x^3-rx^2+qx-p=0$$

be in harmonic progression, show that the mean root is $3p/q$.
[*Allahabad*, 1956]

16. If α, β, γ be the roots of the equation

$$x^3+px^2+qx+r=0,$$

find the value of $\Sigma\alpha^2\beta$. [*Kashmir*, 1954]

17. If α, β, γ are the roots of $x^3-4x^2+2x-1=0$, find the value of $\Sigma\alpha^2\beta$ and $\Sigma\alpha^3$. [*Poona*, 1960]

18. Find the sum of the sixth powers of the roots of the equation $x^3-x-1=0$. [*Nagpur*, 1950]

19. If α, β, γ are the roots of $x^3+3x+9=0$, prove that $\alpha^9+\beta^9+\gamma^9=0$. [*Karnatak*, 1959]

20. Show that the sum of the fourth powers of the roots of $x^5+px^3+qx^2+s=0$ is $2p^2$. [*Madras*, 1954]

CHAPTER XII

TRANSFORMATION OF EQUATIONS

12·1. Transformation of Equations. Without knowing the roots of an equation, we can often transform it into another equation whose roots are related to the roots of the first equation in some way. Such a transformation sometimes helps us in the discussion of the roots of the original equation. We give in the following articles some of the more important transformations.

12·11. Roots with signs changed. Let $\alpha, \beta, \gamma, \ldots$ be the roots of the equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0.$$

Then we have

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n \equiv (x-\alpha)(x-\beta)(x-\gamma)\ldots.$$

Changing x into $-x$, and dividing by $(-1)^n$, we get

$$x^n - p_1 x^{n-1} + p_2 x^{n-2} - \ldots + (-1)^n p_n$$
$$\equiv (x+\alpha)(x+\beta)(x+\gamma)\ldots,$$

showing that the equation whose roots are $-\alpha$, $-\beta$, $-\gamma$, ... is

$$x^n - p_1 x^{n-1} + p_2 x^{n-2} - \ldots + (-1)^n p_n = 0.$$

Hence, to transform an equation into another whose roots are numerically equal to the roots of the given equation but have the opposite sign, *change the sign of every alternate term of the given equation beginning with the second.*

NOTE. If any power of x is missing, care must be taken to suppose that it is present, but has a zero coefficient.

Ex. Obtain the equation whose roots are the roots of

$$x^5-3x^3+6x^2-8=0$$

with their signs changed.

Here the coefficients are 1, 0, -3, 6, 0, -8. Hence changing the sign of every alternate term, the required equation is

$$x^5-3x^3-6x^2+8=0.$$

12·12. Roots multiplied by a given number.

To obtain an equation whose roots are a fixed multiple of the roots of a given equation.

Let the given equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n=0$$

have the roots $\alpha, \beta, \gamma, \ldots$. Then we have to find the equation whose roots are $k\alpha, k\beta, k\gamma, \ldots$ say.

Now the required equation is

$$(x-k\alpha)(x-k\beta)(x-k\gamma)\ldots=0,$$

or, dividing by k^n, $\left(\frac{x}{k}-\alpha\right)\left(\frac{x}{k}-\beta\right)\ldots=0.$

This can be obtained from $(x-\alpha)(x-\beta)\ldots=0$ by writing x/k for x.

Hence the required equation can be obtained from $x^n+p_1x^{n-1}+\ldots=0$ also by writing x/k for x. It must, therefore, be

$$\left(\frac{x}{k}\right)^n+p_1\left(\frac{x}{k}\right)^{n-1}+\ldots+p_n=0.$$

Multiplying out by k^n, we see that the required equation is

$$x^n+p_1kx^{n-1}+p_2k^2x^{n-2}+\ldots+p_nk^n=0,$$

i.e., to obtain the new equation, *multiply the successive coefficients of the given equation by*

$$1,\ k,\ k^2,\ k^3,\ \ldots,\ k^n.$$

12·13. Reciprocal Roots. *To obtain an equation whose roots are reciprocals of the roots of a given equation.*

Let the given equation be

$$x^n+p_1x^{n-1}+\ldots+p_{n-1}x+p_n=0, \qquad (1)$$

and let $\alpha, \beta, \gamma, \ldots$ be its roots. Then the required equation is

$$\left(x-\frac{1}{\alpha}\right)\left(x-\frac{1}{\beta}\right)\left(x-\frac{1}{\gamma}\right)\ldots=0,$$

or, on multiplying by $\left(\frac{\alpha}{x}\right)\left(\frac{\beta}{x}\right)\left(\frac{\gamma}{x}\right)\ldots$

$$\left(\alpha-\frac{1}{x}\right)\left(\beta-\frac{1}{x}\right)\left(\gamma-\frac{1}{x}\right)\ldots=0,$$

i.e.,
$$\left(\frac{1}{x}-\alpha\right)\left(\frac{1}{x}-\beta\right)\left(\frac{1}{x}-\gamma\right)\ldots=0.$$

This can be obtained from $(x-\alpha)(x-\beta)\ldots=0$ by writing $1/x$ for x.

Hence the required equation can also be obtained from (1) by writing $1/x$ for x. It is, therefore,

$$\left(\frac{1}{x}\right)^n+p_1\left(\frac{1}{x}\right)^{n-1}+\ldots+p_{n-1}\left(\frac{1}{x}\right)+p_n=0,$$

or, on multiplying out by x^n,

$$p_nx^n+p_{n-1}x^{n-1}+\ldots+p_1x+1=0.$$

12·14. To diminish the roots by a given number. *To obtain an equation whose roots are equal to the roots of a given equation diminished by a fixed number.*

Let α, β, ... be the roots of the given equation $f(x)=0$. If we write $x+h$ for x in $f(x)=0$, i.e., in

$$(x-\alpha)(x-\beta)\ldots=0,$$

we get

$$(x+h-\alpha)(x+h-\beta)\ldots=0,$$

the roots of which are $\alpha-h$, $\beta-h$, ... Therefore, to diminish the roots by h we have only to *write $x+h$ for x in the given equation.*

For quick numerical computation, we notice that if the new equation be

$$A_0x^n+A_1x^{n-1}+\ldots+A_n=0, \quad \ldots \quad (1)$$

the original equation must have been

$$A_0(x-h)^n+A_1(x-h)^{n-1}+\ldots+A_n=0, \quad (2)$$

because writing $x+h$ for x in (2) gives (1).

If we divide the left-hand side of (2) by $x-h$, the remainder will be evidently A_n, and A_n can thus be determined.

The quotient will be $A_0(x-h)^{n-1}+A_1(x-h)^{n-2}+\ldots+A_{n-1}$, which on division by $x-h$ will leave a remainder A_{n-1}, and thus A_{n-1} gets determined.

Thus *the coefficients A_n, A_{n-1}, A_{n-2}, ... of the transformed equation can all be determined by repeatedly dividing the given equation by $x-h$.*

At first sight it appears that this method would be tedious, but if only the coefficients are written, and the work is properly arranged as in § 11·2, it is quicker than writing $x+h$ in the original equation, expanding each term, and rearranging.

This method is extensively employed in solving numerical equations, as the student will see later.

NOTE. *To increase the roots* of a given equation by h we have simply to diminish the roots by $-h$.

Ex. Find the equation whose roots are equal to the roots of

$$x^4+x^3-3x^2-x+2=0$$

each diminished by 3. [*Allahabad*, 1956]

Dividing repeatedly by $x-3$ and writing only the coefficients, we have

1	1	-3	-1	2
	3	12	27	78
	4	9	26	80
	3	21	90	
	7	30	116	
	3	30		
	10	60		
	3			
	13			

Hence the transformed equation is

$$x^4+13x^3+60x^2+116x+80=0.$$

EXPLANATION. The coefficients of the given polynomial are written in the first line. Division by $x-3$ gives 80 as the remainder and $x^3+4x^2+9x+26$ as the quotient. Division by $x-3$ again gives 116 as the remainder and $x^2+7x+30$ as the quotient, and so on. The successive remainders 80, 116, ... are the coefficients in the transformed equation beginning from the last one.

It is *not* necessary to make the coefficient of x^n unity before applying this method.

12·15. Removal of terms. *To remove the second term of an equation.* If we diminish by h the roots of the equation

$$a_0x^n+a_1x^{n-1}+\ldots+a_n=0,$$

the transformed equation is

$$a_0(x+h)^n+a_1(x+h)^{n-1}+\ldots=0,$$

i.e., $$a_0x^n+(a_1+nha_0)x^{n-1}+\ldots=0.$$

Now if we choose h to be such that

$$a_1+nha_0=0,$$

the coefficient of x^{n-1} will vanish. Hence *the second term of the given equation can be removed by diminishing its roots by* $-a_1/na_0$.

Similarly any other term can be removed, but then h will have to be found from an equation of a higher degree.

For example, to remove the third term, h must be such that

$$a_2+(n-1)ha_1+\tfrac{1}{2}n(n-1)h^2a_0=0.$$

Ex. 1. Remove the second term from the equation

$$x^3+6x^2-7x-4=0.$$

The roots have to be diminished by $-a_1/na_0$, i.e., by $-6/3$, or -2. Hence the coefficients in the transformed equation can be obtained by dividing x^3+6x^2-7x-4 repeatedly by $x+2$. The coefficients are as follows

$$\begin{array}{cccc}
1 & 6 & -7 & -4 \\
 & -2 & -8 & 30 \\
\cline{2-4}
 & 4 & -15 & 26 \\
 & -2 & -4 & \\
\cline{2-3}
 & 2 & -19 & \\
 & -2 & & \\
\cline{2-2}
 & 0 & &
\end{array}$$

We see that the transformed equation is

$$x^3-19x+26=0.$$

Ex. 2. Remove the second term from the cubic equation

$$ax^3+3bx^2+3cx+d=0.$$

Putting $x=y+h$, the equation becomes

$$a(y+h)^3+3b(y+h)^2+3c(y+h)+d=0,$$

or $ay^3+3(ah+b)y^2+3(ah^2+2bh+c)y+(ah^3+3bh^2+3ch+d)=0.$

If we put $h=-b/a$, the second term vanishes, and the transformed equation becomes

$$ay^3+3(c-b^2/a)y+(d-3bc/a+2b^3/a^2)=0.$$

12·16. Transformation in general. Let $f(x)=0$ be a given equation; and let it be required to obtain a new equation, in y say, whose roots are connected with the roots of the given equation by a relation of the form $\phi(x, y)=0$. The transformed equation can be obtained by eliminating x between $f(x)=0$ and $\phi(x, y)=0$, for this will give the equation satisfied by y.

Ex. 1. Find the equation whose roots are the squares of the roots of

$$x^4+x^3+2x^2+x+1=0. \qquad (1)$$

If x is a root of the above equation and y a root of the required equation, then

$$y=x^2, \text{ i.e., } x=\sqrt{y}.$$

Substituting in (1), we get

$$y^2+y\sqrt{y}+2y+\sqrt{y}+1=0,$$

or
$$(y^2+2y+1)^2=\{(y+1)\sqrt{y}\}^2,$$

or
$$y^4+3y^3+4y^2+3y+1=0.$$

This is the required equation.

Ex. 2. If α, β, γ be the roots of the equation

$$x^3+px^2+qx+r=0,$$

find the equation whose roots are

$$\beta^2+\beta\gamma+\gamma^2,\ \gamma^2+\gamma\alpha+\alpha^2,\ \alpha^2+\alpha\beta+\beta^2.$$

Let y be a root of the required equation, then

$$\begin{aligned} y &= \beta^2+\beta\gamma+\gamma^2=(\alpha^2+\beta^2+\gamma^2)-\alpha^2+\alpha\beta\gamma/\alpha \\ &= \{(\alpha+\beta+\gamma)^2-2\Sigma\alpha\beta\}-\alpha^2+\alpha\beta\gamma/\alpha \\ &= (p^2-2q)-\alpha^2-r/\alpha, \end{aligned}$$

or
$$\alpha y=(p^2-2q)\alpha-\alpha^3-r. \qquad . \quad . \quad . \quad (1)$$

But α is a root of the given equation; therefore

$$0=\alpha^3+p\alpha^2+q\alpha+r. \qquad . \quad . \quad . \quad (2)$$

The required equation will be obtained by eliminating α between (1) and (2).

From (1) and (2), by addition,

$$\alpha y=(p^2-q)\alpha+p\alpha^2. \quad . \quad . \quad . \quad (3)$$

As $\alpha\neq 0$, (3) gives $\quad \alpha=(y+q-p^2)/p$.

Substituting this value of α in (2), we get

$$\frac{(y+q-p^2)^3}{p^3}+\frac{(y+q-p^2)^2}{p}+\frac{q(y+q-p^2)}{p}+r=0,$$

or, on simplification,

$$y^3+(3q-2p^2)y^2+(3q^2-3qp^2+p^4)y+(q^3-q^2p^2+rp^3)=0.$$

This is the required equation.

12·2. Equation of Squared Differences of a Cubic.

To find the equation whose roots are the squares of the differences of the roots of a given cubic equation.

Let the given cubic be

$$x^3+qx+r=0, \quad . \quad . \quad . \quad (1)$$

and let α, β, γ be its roots. It is required to find the equation whose roots are $(\alpha-\beta)^2$, $(\beta-\gamma)^2$, $(\gamma-\alpha)^2$. Writing y for any one of these roots, we see that

$$y=(\beta-\gamma)^2=\alpha^2+\beta^2+\gamma^2-\alpha^2-2\alpha\beta\gamma/\alpha.$$

But $\quad \alpha^2+\beta^2+\gamma^2=-2q$(§ 11·52, Ex. 1) and $\alpha\beta\gamma=-r$.

Hence the relation between the roots is

$$y=-2q-x^2+2r/x,$$

or

$$x^3+(y+2q)x-2r=0. \quad . \quad . \quad . \quad (2)$$

Subtracting (1) from (2), we get

$$(y+q)x-3r=0, \text{ or } x=3r/(y+q).$$

Substituting this value of x in (1), and simplifying, we get

$$y^3+6qy^2+9q^2y+4q^3+27r^2=0 \quad . \quad . \quad . \quad (3)$$

as the required equation.

We would have evidently got the same equation for y if we had initially assumed y to be equal to $(\alpha-\beta)^2$ or $(\gamma-\alpha)^2$.

If we have to find the equation of squared differences for a cubic in which the second term is not zero, we must first remove its second term as in Ex. 2, § 12·15.

12·3. Reciprocal Equations. An equation which is unaltered by changing x into $1/x$ is called a ***reciprocal equation.***

Let an equation in x be

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n=0. \qquad (1)$$

To find the relations which must subsist between the coefficients in order that this may be a reciprocal equation, change x to $1/x$. Then the equation becomes

$$p_nx^n+p_{n-1}x^{n-1}+p_{n-2}x^{n-2}+\ldots+p_1x+1=0. \qquad (2)$$

If this be same as (1), we must have

$$\frac{p_{n-1}}{p_n}=p_1,\ \frac{p_{n-2}}{p_n}=p_2,\ \ldots,\ \frac{1}{p_n}=p_n. \quad . \quad . \quad . \quad (3)$$

The last equation of (3) gives $p_n{}^2=1$, or $p_n=\pm 1$. If $p_n=1$, (3) gives

$$p_{n-1}=p_1,\ \ p_{n-2}=p_2, \ldots;$$

while if $p_n=-1$, we get

$$p_{n-1}=-p_1, p_{n-2}=-p_2, \ \ldots .$$

Thus in a reciprocal equation the coefficients of terms equidistant from the beginning and the end are either

(i) equal both in magnitude and sign,

or (ii) equal in magnitude but opposite in sign.

It is clear from the definition that if α be a root of a reciprocal equation, $1/\alpha$ must also be a root. Hence the roots of a reciprocal equation occur in pairs: α, $1/\alpha$; β, $1/\beta$; etc. When the degree of the equation is odd there must be a root which is its own reciprocal (i.e., 1 or -1). Substitution will show that for an odd degree equation of type (i), this root is -1; while for an odd degree equation of type (ii), it is 1. Division by $x+1$ or $x-1$, as the case may be, leaves a reciprocal equation of type (i) of an even degree.

A reciprocal equation of type (ii), of an even degree $2m$, may be written as

$$x^{2m}-1+p_1x(x^{2m-2}-1)+\ldots=0.$$

Evidently this is satisfied both by 1 and -1. Division by x^2-1 leaves in this case also an equation of type (i) of an even degree.

Thus all reciprocal equations can be reduced, by division by $(x-1)$, $(x+1)$ or (x^2-1), to equations of type (i) of even degree. The degree of such reciprocal equations can then be reduced to half by the substitution $x+1/x=y$, as illustrated in the example below.

Ex. Solve the equation

$$x^5-5x^4+9x^3-9x^2+5x-1=0. \quad \text{[\textit{Banaras}, 1960]}$$

This is a reciprocal equation of the second type. We see by substitution that $x=1$ is a root. Dividing out by $x-1$, the given equation becomes

$$x^4-4x^3+5x^2-4x+1=0,$$

which is a reciprocal equation of the first type. Dividing by x^2, we get

$$x^2-4x+5-4(1/x)+1/x^2=0. \qquad (1)$$

Put $x+1/x=y$; then $x^2+1/x^2=y^2-2$; and (1) becomes

$$y^2-4y+3=0.$$

Therefore $\qquad y=1$ or 3.

Solving the equations

$$x+1/x=1 \text{ and } x+1/x=3,$$

we get $\qquad x=\frac{1}{2}\pm i(\frac{1}{2}\sqrt{3}),\ \frac{3}{2}\pm\frac{1}{2}\sqrt{5}.$

Hence the roots of the given equation are

$$1,\ \tfrac{1}{2}+i(\tfrac{1}{2}\sqrt{3}),\ \tfrac{1}{2}-i(\tfrac{1}{2}\sqrt{3}),\ \tfrac{3}{2}+\tfrac{1}{2}\sqrt{5},\ \tfrac{3}{2}-\tfrac{1}{2}\sqrt{5}.$$

Examples on Chapter XII

1. Find the equation whose roots are the roots of

$$x^7+3x^5+x^3-x^2+7x+2=0$$

with their signs changed.

2. Find the equation whose roots are three times the roots of

$$x^3+2x^2-4x+1=0.$$

3. Find the equation whose roots are twice the reciprocals of the roots of

$$x^4+3x^3-6x^2+2x-4=0.$$

4. **Diminish by 3 the roots of the equation**

$$x^3-9x^2+28x-27=0.$$

5. Find the equation whose roots are the roots of

$$x^5+4x^3-x^2+11=0,$$

each diminished by 3. [*Allahabad*, 1960]

6. Find the equation each of whose roots is greater by unity than a root of $x^3-5x^2+6x-3=0$. [*Sagar*, 1948]

Transform into an equation lacking the second term :

7. $x^3-6x^2+4x-7=0$. [*Delhi*, 1958]

8. $x^4+8x^3+x-5=0$. [*Kashmir*, 1954]

9. $x^4+4x^3+2x^2-4x-2=0$.

10. Transform the equation

$$x^4-4x^3-18x^2-3x+2=0$$

into one from which the third term is missing.
[*Allahabad*, 1960]

11. Find the equation whose roots are the cubes of the roots of

$$x^3+3x^2+2=0.$$

If α, β, γ are the roots of $x^3+qx+r=0$, form the equation whose roots are

12. $\beta+\gamma,\ \gamma+\alpha,\ \alpha+\beta$. 13. $\beta^2\gamma^2,\ \gamma^2\alpha^2,\ \alpha^2\beta^2$.

14. $\beta\gamma/\alpha,\ \gamma\alpha/\beta,\ \alpha\beta/\gamma$.

15. $1/\beta+1/\gamma,\ 1/\gamma+1/\alpha,\ 1/\alpha+1/\beta$. [*Delhi*, 1949]

16. $(\beta+\gamma)/\alpha^2,\ (\gamma+\alpha)/\beta^2,\ (\alpha+\beta)/\gamma^2$.

17. $\beta\gamma+1/\alpha,\ \gamma\alpha+1/\beta,\ \alpha\beta+1/\gamma$. [*Sagar*, 1950]

If α, β, γ be the roots of the equation $x^3+px^2+qx+r=0$, find the equation whose roots are

18. $\alpha(\beta+\gamma),\ \beta(\gamma+\alpha),\ \gamma(\alpha+\beta)$. [*Aligarh*, 1952]

19. $\alpha/(\beta+\gamma),\ \beta/(\gamma+\alpha),\ \gamma/(\alpha+\beta)$. [*Alld.*, 1959]

20. $\alpha+1/\beta\gamma,\ \beta+1/\gamma\alpha,\ \gamma+1/\alpha\beta$. [*I.A.S.*, 1960]

21. If α, β, γ are the roots of $2x^3+3x^2-x-1=0$, find the equation whose roots are

$$1/(1-\alpha),\ 1/(1-\beta),\ 1/(1-\gamma).$$ [*Aligarh*, 1960]

22. If α, β, γ be the roots of $x^3+qx+r=0$, prove that the equation whose roots are :

$$\frac{\beta}{\gamma}+\frac{\gamma}{\beta},\ \frac{\gamma}{\alpha}+\frac{\alpha}{\gamma},\ \frac{\alpha}{\beta}+\frac{\beta}{\alpha}$$

is $$r^2(x+1)^3+q^3(x+1)+q^3=0.$$

23. Solve the equation

$$x^3+6x^2+12x-19=0,$$

by removing its second term. [*Nagpur*, 1954]

24. Form the equation of the squared differences of the cubic

$$x^3-7x+6=0.$$ [*Allahabad*, 1960]

25. Form the equation of the squared differences of

$$x^3+6x^2+7x+2=0.$$

Solve the equations :

26. $2x^4-5x^3+4x^2-5x+2=0.$

27. $x^4-10x^3+26x^2-10x+1=0.$ [*Nagpur*, 1953]

28. $2x^5-7x^4-x^3-x^2-7x+2=0.$ [*I.A.S.*, 1956]

29. $2x^5-15x^4+37x^3-37x^2+15x-2=0.$

30. $6x^6-25x^5+31x^4-31x^3+25x-6=0.$ [*Mysore*, 1951]

31. Show that the equation

$$x^4-3x^3+4x^2-2x+1=0$$

can be transformed into a reciprocal equation by diminishing the roots by unity. Hence solve the equation. [*Alld.*, 1959]

32. Transform the equation

$$x^4+3x^3+kx^2+12x+16=0$$

into a reciprocal equation and solve it when $k=4$.

[*I.A.S.*, 1957]

CHAPTER XIII

CUBIC AND BIQUADRATIC EQUATIONS

13·1. Algebraic Solutions. The various methods devised for solving cubic and biquadratic equations algebraically are based on one of the following artifices :

(i) Assuming for the root a general form involving radicals;

(ii) Resolving the expression into factors;

(iii) Obtaining the values of linear functions of the roots.

Some of these methods are given below.

It may be mentioned that these methods are generally unsuited for numerical computation. Numerical equations are best solved by the methods of the next chapter.

13·2. Solution of the Cubic. Let the cubic equation be

$$ax^3+3bx^2+3cx+d=0. \quad . \quad . \quad (1)$$

Removing the second term by putting $x=y-b/a$, this becomes (Ex. 2, § 12·15)

$$ay^3+3(c-b^2/a)y+(d-3bc/a+2b^3/a^2)=0.$$

Multiplying the roots of this equation by a to avoid fractions and writing z for ay, it transforms into

$$z^3+3Hz+G=0, \quad . \quad . \quad . \quad (2)$$

where,

$$z\equiv ay\equiv ax+b,\ H\equiv ac-b^2,\ G\equiv a^2d-3abc+2b^3.$$

To solve this assume that the roots are of the form $\sqrt[3]{p}+\sqrt[3]{q}$, where p and q are to be determined. Putting

$$z=\sqrt[3]{p}+\sqrt[3]{q} \quad . \quad . \quad . \quad (3)$$

and cubing, we get

$$z^3=p+q+3\,.\sqrt[3]{p}\sqrt[3]{q}(\sqrt[3]{p}+\sqrt[3]{q}),$$

i.e.,
$$z^3-3(\sqrt[3]{p}\sqrt[3]{q})z-(p+q)=0. \qquad (4)$$

Comparing the coefficients in (2) and (4), we get

$$\sqrt[3]{p}\sqrt[3]{q}=-H,\ p+q=-G, \quad . \quad . \quad (5)$$

i.e.,
$$p+q=-G,\ pq=-H^3.$$

These equations determine p and q; and so the roots (3) get determined.

We see that p and q are the roots of the quadratic

$$t^2+Gt-H^3=0.$$

Solving it, we get

$$\left.\begin{aligned} p&=\tfrac{1}{2}[-G+\sqrt{(G^2+4H^3)}], \\ \text{and} \quad q&=\tfrac{1}{2}[-G-\sqrt{(G^2+4H^3)}]. \end{aligned}\right\} \quad . \quad . \quad (6)$$

Substitution of these values in (3) gives a root of equation (2), and, therefore, of (1).

We notice that $\sqrt[3]{p}$ has three values, viz. m, ωm and $\omega^2 m$, where m is a cube root of p and ω is one of the imaginary cube roots of unity. But we cannot take the three values of $\sqrt[3]{q}$ independently, for by (5), $\sqrt[3]{q}=-H/\sqrt[3]{p}$. Hence

$$z=\sqrt[3]{p}-H/\sqrt[3]{p}, \quad . \quad . \quad . \quad (7)$$

so that corresponding to each value of $\sqrt[3]{p}$ we get only one value of z.

Since $z \equiv ax+b$, i.e., $x \equiv (z/a)-b$, so the roots of (1) are

$$\frac{1}{a}\left(m-\frac{H}{m}\right)-b,\ \frac{1}{a}\left(\omega m-\omega^2\frac{H}{m}\right)-b,$$

and $$\frac{1}{a}\left(\omega^2 m-\omega\frac{H}{m}\right)-b, \tag{8}$$

where $m=\sqrt[3]{[\tfrac{1}{2}\{-G+\sqrt{(G^2+4H^3)}\}]}$.

This solution of the cubic is generally known as Cardan's solution*.

13·3. Nature of the Roots of a Cubic. Let α, β, γ be the roots of the cubic

$$ax^3+3bx^2+3cx+d=0. \quad . \quad . \quad . \quad (1)$$

Then, the roots of the transformed equation

$$z^3+3Hz+G=0, \quad . \quad . \quad . \quad (2)$$

where H and G have the same meanings as in the preceding article, are $a\alpha+b$, $a\beta+b$, $a\gamma+b$.

The equation whose roots are the squares of the differences of the roots of (2), is (§ 12·2)

$$y^3+18Hy^2+81H^2y+27(G^2+4H^3)=0. \tag{3}$$

Its roots are, therefore,

$$a^2(\beta-\gamma)^2,\ a^2(\gamma-\alpha)^2,\ a^2(\alpha-\beta)^2. \tag{4}$$

*Named after the Italian scholar Hieronimo Cardano (1501-1576). The solution was really discovered by Tartaglia (1499-1557), from whom Cardan obtained it after many solicitations and after giving the most solemn and sacred promises of secrecy. But Cardan broke his most solemn vows and published the solution in his *Ars Magna*, thus causing much chagrin and disappointment to Tartaglia. It is possible that the solution of the cubic was discovered prior to Tartaglia by Scipione del Ferro (1465-1526), but no record exists of his work—Cajori: *A History of Mathematics*.

The product of these roots is equal to the constant term in (3) with its sign changed, i.e.,

$$a^6(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2=-27(G^2+4H^3). \quad (5)$$

The nature of the roots α, β, γ can be obtained by a consideration of this product.

Since imaginary roots occur in pairs, therefore equation (1) will have either all real roots, or one real and two imaginary roots. The following cases can occur:

(i) The roots α, β, γ are all real and different. In this case $a^6(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2$ is positive. Therefore, by (5), G^2+4H^3 is negative.

(ii) One root, say α, is real and the other two imaginary. Let β and γ be $m\pm in$. Then

$$\begin{aligned} &a^6(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2 \\ &\quad=a^6(2in)^2(m-in-\alpha)^2(m+in-\alpha)^2 \\ &\quad=-4a^6n^2\{(m-\alpha)^2+n^2\}^2, \end{aligned}$$

which is negative, whatever a, α, m, n may be. Therefore, by (5), G^2+4H^3 is positive in this case.

(iii) Two of the roots, say β, γ, are equal. Then $a^6(\beta-\gamma)^2(\gamma-\alpha)^2(\alpha-\beta)^2$, and therefore G^2+4H^3, is zero.

(iv) α, β, γ are all equal. In this case all the three roots of equation (3) are zero. This will be so if $H=G=0$.

Conversely, it is easy to see that

I. *When G^2+4H^3 is negative, the roots of the cubic are all real;*

II. *When G^2+4H^3 is positive, the cubic has two imaginary roots;*

III. *When* $G^2+4H^3=0$, *the cubic has two equal roots;* and

IV. *When* $G=H=0$, *all the roots of the cubic are equal.*

The above criteria determine the nature of the roots.

On substituting the values of G and H, it can be seen that

$$G^2+4H^3 \equiv a^2\{a^2d^2-6abcd+4ac^3+4b^3d-3b^2c^2\}.$$

The expression in brackets is called the *discriminant* of the cubic (1), and is denoted by Δ. It is evident that the discriminant of equation (2) is G^2+4H^3 itself. The discriminant of an equation is an expression in terms of the coefficients which vanishes when the equation has equal roots.

13·31. Nature of the Roots from Cardan's Solution. The above results can also be inferred from Cardan's solution of the cubic. We have seen in § 13·2 that a cubic can be transformed into the equation $z^3+3Hz+G=0$, whose roots are the three values of

$$\sqrt[3]{p}+\sqrt[3]{q}, \quad . \quad . \quad . \quad (1)$$

where

$$p=\tfrac{1}{2}[-G+\sqrt{(G^2+4H^3)}],\ q=\tfrac{1}{2}[-G-\sqrt{(G^2+4H^3)}], \quad (2)$$

and only such values of the cube roots of p and q are taken which satisfy

$$\sqrt[3]{p}\sqrt[3]{q}=-H. \quad . \quad . \quad . \quad (3)$$

When $G^2+4H^3=0$, we see from (2) that $p=q=-\tfrac{1}{2}G$. If m denotes the real cube root of p, the roots (1) in this case become

$$m+m,\ \omega m+\omega^2 m,\ \omega^2 m+\omega m,$$

i.e.,
$$2m,\ -m,\ -m.$$

Thus two roots are equal when $\Delta=0$.

When G^2+4H^3 is positive, we see from (2) that p and q are real and unequal. If m and n denote respectively the real cube roots of p and q, the values of (1) are

$$m+n,\ \omega m+\omega^2 n,\ \omega^2 m+\omega n.$$

So out of the three roots of the cubic, one is real while the other two are imaginary.

When G^2+4H^3 is negative, p is of the form $u+iv$ and q is of the form $u-iv$. Put

$$u=r\cos\theta,\ v=r\sin\theta;$$

then, by De Moivre's theorem,

$$\sqrt[3]{p}=\{r\cos\theta+ir\sin\theta\}^{1/3}$$
$$=\sqrt[3]{r}\{\cos\tfrac{1}{3}(\theta+2k\pi)+i\sin\tfrac{1}{3}(\theta+2k\pi)\}, \qquad (4)$$

and
$$\sqrt[3]{q}=\sqrt[3]{r}\{\cos\tfrac{1}{3}(\theta+2k\pi)-i\sin\tfrac{1}{3}(\theta+2k\pi)\}, \qquad (5)$$

where $k=0$, 1, or 2.

Adding (4) and (5) we get the solution.

The same value of k should be taken in both (4) and (5), as by (3), $\sqrt[3]{p}\sqrt[3]{q}$ has to be a real quantity.

Thus the three roots are

$$\sqrt[3]{r}\,.\,2\cos\tfrac{1}{3}(\theta+2k\pi),\ k=0,1,2.$$

Hence all the roots are real when G^2+4H^3 is negative.

13·32. Application of Cardan's method to numerical examples. The method of Cardan can be employed to solve numerical examples as illustrated in the examples below. The first two examples have been specially framed to give convenient solutions. The evaluation of p and $\sqrt[3]{p}$ both involve extraction of roots, and in general tables will be necessary for their evaluation. The third example illustrates this point. In such cases it is more convenient to find the real roots by the methods of the next chapter.

Ex. 1. Solve the equation

$$x^3-3x+1=0.$$

If we assume $x=\sqrt[3]{p}+\sqrt[3]{q}$, then

$$\sqrt[3]{p}\sqrt[3]{q}=1,\ p+q=-1.$$

Therefore $\quad p-q=\sqrt{\{(p+q)^2-4pq\}}=\sqrt{(1-4)}=i\sqrt{3},$

so that $p=(-1+i\sqrt{3})/2=\cos 120°+i\sin 120°.$

Hence $\sqrt[3]{p}=\cos\frac{1}{3}(120+360\,k)°+i\sin\frac{1}{3}(120+360\,k)°,$

and similarly $\sqrt[3]{q}=\cos\frac{1}{3}(120+360\,k)°-i\sin\frac{1}{3}(120+360\,k)°.$

Hence the roots are

$$2\cos 40°,\ 2\cos 160°,\ 2\cos 280°,$$

i.e., $$1{\cdot}532,\ -1{\cdot}879,\ 0{\cdot}347.$$

Ex. 2. Solve by Cardan's method the cubic

$$35x^3-18x^2+1=0.$$

Since the third term is missing from this equation, it may be transformed into an equation lacking the second term very conveniently by putting $x=1/y$. The equation in y is

$$y^3-18y+35=0. \qquad . \quad . \quad . \quad (1)$$

Assume a solution $y=\sqrt[3]{p}+\sqrt[3]{q}$. $\qquad . \quad . \quad . \quad (2)$

Then $p+q=-35,\ \sqrt[3]{p}\sqrt[3]{q}=6$ i.e. $pq=216$. $\qquad . \quad . \quad . \quad (3)$

We see that p and q are the roots of the equation

$$t^2+35t+216=0, \text{ or } (t+8)(t+27)=0.$$

Therefore $p=-8$ and $q=-27$.

Hence $\sqrt[3]{p}=-2,\ -2\omega,$ or $-2\omega^2$;

and, by (3), the corresponding values of $\sqrt[3]{q}$ are

$$\sqrt[3]{q}=6/\sqrt[3]{p}=-3,\ -3\omega^2,\ -3\omega, \text{ respectively.}$$

So the roots of (1) are

$$-5,\ -2\omega-3\omega^2 \text{ and } -2\omega^2-3\omega.$$

The roots of the given equation are the reciprocals of these.

Now $$-2\omega-3\omega^2=-2\left(\frac{-1+i\sqrt{3}}{2}\right)-3\left(\frac{-1-i\sqrt{3}}{2}\right)$$

$$=\frac{5+i\sqrt{3}}{2}.$$

Its reciprocal $=\dfrac{2}{5+i\sqrt{3}}=\dfrac{2(5-i\sqrt{3})}{25+3}=\frac{1}{14}(5-i\sqrt{3}).$

$1/(-2\omega^2-3\omega)$ can be found similarly. Hence the required roots are

$$-\tfrac{1}{5},\ \tfrac{1}{14}(5-i\sqrt{3}),\ \tfrac{1}{14}(5+i\sqrt{3}).$$

Ex. 3. Solve by Cardan's method the equation

$$x^3-6x-13=0.$$

Assume a solution $y=\sqrt[3]{p}+\sqrt[3]{q}$. . . . (1)

Then $$p+q=13,\ \sqrt[3]{p}\sqrt[3]{q}=2. \quad . \quad . \quad . \quad (2)$$

We see that p and q are the roots of $t^2-13t+8=0$.

Therefore $\quad p=\frac{1}{2}(13+\sqrt{137}),\ q=\frac{1}{2}(13-\sqrt{137})$.

Substituting in (1), and taking only such combinations which make $\sqrt[3]{p}\sqrt[3]{q}$ real as required by (2), we see that the roots of the given equation are

$$\sqrt[3]{(\tfrac{13}{2}+\tfrac{1}{2}\sqrt{137})}+\sqrt[3]{(\tfrac{13}{2}-\tfrac{1}{2}\sqrt{137})},$$

$$\omega\sqrt[3]{(\tfrac{13}{2}+\tfrac{1}{2}\sqrt{137})}+\omega^2\sqrt[3]{(\tfrac{13}{2}-\tfrac{1}{2}\sqrt{137})},$$

and $$\omega^2\sqrt[3]{(\tfrac{13}{2}+\tfrac{1}{2}\sqrt{137})}+\omega\sqrt[3]{(\tfrac{13}{2}-\tfrac{1}{2}\sqrt{137})}.$$

[We can find from Mathematical tables that

$$\sqrt{137}=11{\cdot}7047.$$

Therefore $p=\frac{13}{2}+\frac{1}{2}\sqrt{(137)}=6{\cdot}5+5{\cdot}8524=12{\cdot}3524$,

and $\quad q=\frac{13}{2}-\frac{1}{2}\sqrt{(137)}=6{\cdot}5-5{\cdot}8524=0{\cdot}6476$.

From the tables $\sqrt[3]{p}=2{\cdot}3117$ and $\sqrt[3]{q}=0{\cdot}8586$.

Hence the real root$=2{\cdot}3117+0{\cdot}8586=3{\cdot}1703$.

The other roots will need some further calculation.]

13·4. Expressing the Cubic as a Difference of two Cubes. We shall now give a second method for solving the cubic.

$$ax^3+3bx^2+3cx+d=0.$$

Let this be reduced by the substitution $z=ax+b$ to the form

$$z^3+3Hz+G=0.$$

Assume that

$$z^3+3Hz+G\equiv\frac{1}{m-n}\{m(z+n)^3-n(z+m)^3\} \qquad (1)$$

$$\equiv z^3-3mnz-mn(m+n),$$

where m, n are quantities to be determined.

Comparing coefficients, we get

$$mn=-H,\ \ mn(m+n)=-G;$$

therefore $\qquad m+n=G/H,\ \ m-n=\sqrt{(G^2+4H^3)}/H,$

which give m and n.

Now since $k^3-l^3=(k-l)(\omega k-\omega^2 l)(\omega^2 k-\omega l)$, so by (1), the cubic $(m-n)(z^3+3Hz+G)=0$ can be factorised as

$$\{\sqrt[3]{m}(z+n)-\sqrt[3]{n}(z+m)\}\{\omega\sqrt[3]{m}(z+n)-\omega^2\sqrt[3]{n}(z+m)\}$$
$$\times\{\omega^2\sqrt[3]{m}(z+n)-\omega\sqrt[3]{n}(z+m)\}=0.$$

Each of the factors equated to zero will give one root of the cubic.

13·5. Solution by Symmetric Functions of Roots.

Let α, β, γ be the roots of the cubic

$$ax^3+3bx^2+3cx+d=0, \quad . \ . \ . \ \ (1)$$

and consider the expression

$$\tfrac{1}{3}\{(\alpha+\beta+\gamma)+k(\alpha+\omega\beta+\omega^2\gamma)+k^2(\alpha+\omega^2\beta+\omega\gamma)\}.$$

When $k=1$, its value is α; when $k=\omega$, its value is γ; and when $k=\omega^2$, its value is β.

We know that

$$\alpha+\beta+\gamma=-3b/a.$$

It remains to find the values of

$$M\equiv\alpha+\omega\beta+\omega^2\gamma \text{ and } N\equiv\alpha+\omega^2\beta+\omega\gamma.$$

Now we may easily show that (cf. ex. 11, p. 199, and ex. 16, p. 211)

$$M^3+N^3=2\Sigma\alpha^3-3\Sigma\alpha^2\beta+12\alpha\beta\gamma$$
$$=2(-27b^3/a^3+27bc/a^2-3d/a)-3(3d/a-9bc/a^2)-12d/a$$
$$=-27(2b^3-3abc+a^2d)/a^3=-27G/a^3,$$

and $MN=\Sigma\alpha^2-\Sigma\alpha\beta=(9b^2/a^2-6c/a)-3c/a$

$$=9(b^2-ac)/a^2=-9H/a^2.$$

Therefore M^3 and N^3 are the roots of the quadratic

$$t^2+\frac{3^3}{a^3}Gt-\frac{3^6}{a^6}H^3=0.$$

Hence, M and N are

$$\frac{3}{a}\left\{\frac{-G\pm\sqrt{(G^2+4H^3)}}{2}\right\}^{1/3}.$$

The three roots of the cubic (1) are then

$$-b/a+\tfrac{1}{3}(M+N),\ -b/a+\tfrac{1}{3}(\omega M+\omega^2 N),\ -b/a+\tfrac{1}{3}(\omega^2 M+\omega N).$$

13·6. Solution of the Biquadratic. Let the biquadratic equation be

$$ax^4+4bx^3+6cx^2+4dx+e=0. \quad . \quad . \quad (1)$$

Assume that the left-hand side multiplied by a (to make the coefficients of x^4 equal in the two expressions) is identically equal to

$$(ax^2+2bx+c+2a\theta)^2-(2Mx+N)^2. \qquad (2)$$

Comparing coefficients, we obtain the following equations to determine M, N and θ:

$$\left.\begin{array}{c} M^2=b^2-ac+a^2\theta,\ MN=bc-ad+2ab\theta, \\ N^2=(c+2a\theta)^2-ae. \end{array}\right\} \qquad (3)$$

Eliminating M and N between these equations, we have

$$4a^3\theta^3-(ae-4bd+3c^2)a\theta+ace+2bcd-ad^2$$
$$-eb^2-c^3=0. \quad (4)$$

We should, therefore, determine first a value of θ from (4), which is known as the *reducing cubic*; next we should determine the corresponding values of M and N from the equations (3); then (2) would enable us to factorise the left hand side of (1), and thus we shall get the following two quadratic equations to solve.

$$\left.\begin{array}{l} ax^2+2(b-M)x+c+2a\theta-N=0, \\ ax^2+2(b+M)x+c+2a\theta+N=0. \end{array}\right\} \qquad (5)$$

The roots of these will be the required roots of (1).

This method of solving the biquadratic is generally known as Ferrari's method.*

The reducing cubic gives three values of θ. These do not, however, lead to three different sets of roots for the biquadratic. They only give three different methods of factorising the left-hand side of the biquadratic.

As a cubic is not always easy to solve, Ferrari's method is not, in general, a practicable method of solving numerical equations.

Ex. Solve the equation

$$x^4-4x^3-4x^2-24x+15=0.$$

Assume that this may be written as

$$(x^2-2x+k)^2-(2Mx+N)^2=0. \qquad (1)$$

Comparing the coefficients, we get

$$2k+4-4M^2=-4, \text{ or } k+4=2M^2,$$
$$k^2-N^2=15, \text{ or } k^2-15=N^2,$$
$$-4k-4MN=-24, \text{ or } 6-k=MN.$$

Eliminating M and N, we have

$$(k+4)(k^2-15)=2(6-k)^2,$$

or

$$k^3+2k^2+9k-132=0.$$

A root of this equation is $k=4$. This value gives $M=2$ and $N=1$. Therefore (1) becomes

$$(x^2-2x+4)^2-(4x+1)^2=0,$$

*Ludovico Ferrari (1522-1560), a man of humble birth, was taken into Cardan's household as a servant at the age of fifteen. He showed such unusual ability that the latter made him his secretary. After three years of service Ferrari left and took up the work of teaching and later became the professor of mathematics at Bologna. A problem involving the solution of a biquadratic was proposed by a teacher at Brescia to Cardan, who being unable to solve it, gave it to Ferrari. The latter succeeded in finding a solution and this was published, with due credit, by Cardan in his *Ars Magna*.—Smith: *A Source Book in Mathematics*.

i.e., $$(x^2-6x+3)(x^2+2x+5)=0.$$
Hence the required roots are
$$3\pm\sqrt{6},\ -1\pm 2i.$$

13·61. Roots of Biquadratic in terms of M. *To express the roots of a biquadratic equation in terms of the roots of its reducing cubic.*

Let α, β, γ, δ be the roots of the biquadratic
$$ax^4+4bx^3+6cx^2+4dx+e=0. \quad . \quad . \quad . \quad (1)$$
Then the left-hand side of (1) may be factorised into quadratic factors in any one of the following three ways:
$$\left.\begin{array}{l} a\{x^2-(\alpha+\beta)x+\alpha\beta\}\{x^2-(\gamma+\delta)x+\gamma\delta\}, \\ a\{x^2-(\alpha+\gamma)x+\alpha\gamma\}\{x^2-(\beta+\delta)x+\beta\delta\}, \\ a\{x^2-(\alpha+\delta)x+\alpha\delta\}\{x^2-(\beta+\gamma)x+\beta\gamma\}. \end{array}\right\} \quad (2)$$
Let θ_r be any one of the three roots of the reducing cubic, and let the corresponding values of M, N be M_r, N_r. Then by the previous article
$$M_r^2=b^2-ac+a^2\theta_r, M_rN_r=bc-ad+2ab\theta_r, N_r^2=(c+2a\theta_r)^2-ae.$$
As in the previous article the quadratic factors of the left-hand side of (1) are
$$(1/a)\{ax^2+2(b-M_r)x+c+2a\theta_r-N_r\}\{ax^2+2(b+M_r)x+c+2a\theta_r+N_r\}$$
where $r=1$, 2, or 3.

These three sets of factors must be the same as those in (2). Hence comparing the coefficients of x, we get
$$\begin{array}{ll} \alpha+\beta=-2(b-M_1)/a, & \gamma+\delta=-2(b+M_1)/a; \\ \alpha+\gamma=-2(b-M_2)/a, & \beta+\delta=-2(b+M_2)/a; \\ \alpha+\delta=-2(b-M_3)/a, & \beta+\gamma=-2(b+M_3)/a. \end{array}$$
From these we have
$$\left.\begin{array}{l} M_1=\tfrac{1}{4}a(\alpha+\beta-\gamma-\delta), \\ M_2=\tfrac{1}{4}a(\alpha+\gamma-\beta-\delta), \\ M_3=\tfrac{1}{4}a(\alpha+\delta-\beta-\gamma); \\ -b=\tfrac{1}{4}a(\alpha+\beta+\gamma+\delta), \end{array}\right\} \quad . \quad . \quad . \quad (3)$$
and since (the last line above), we can write
$$\begin{array}{l} a\alpha+b=M_1+M_2+M_3, \\ a\beta+b=M_1-M_2-M_3, \\ a\gamma+b=-M_1+M_2-M_3, \\ a\delta+b=-M_1-M_2+M_3. \end{array}$$

These relations give the roots of the biquadratic in terms of the roots of the reducing cubic.

NOTE. Since $M_r=\sqrt{(b^2-ac+a^2\theta_r)}$, we get two values each of M_1, M_2 and M_3. In fact eight combinations can be obtained with $\pm M_1\pm M_2\pm M_3$. However, from (3), we get $M_1M_2M_3=\frac{1}{64}a^3(\Sigma\alpha^3-\Sigma\alpha^2\beta+2\Sigma\alpha\beta\gamma)$, which can be shown to be equal to $-\frac{1}{2}(a^2d-3abc+2b^3)$, i.e. $-\frac{1}{2}G$. Hence only four of the combinations will be admissible.

13·7. Solution by Radicals. Let the biquadratic equation be

$$ax^4+4bx^3+6cx^2+4dx+e=0. \qquad \ldots \quad (1)$$

Removing the second term and multiplying the roots by a to avoid fractions, the equation may be transformed to

$$z^4+6Hz^2+4Gz+a^2I-3H^2=0, \qquad \ldots \quad (2)$$

where $z\equiv ax+b$ and, as can be easily verified,

$$H\equiv ac-b^2,\ G\equiv a^2d-3abc+2b^3,\ I\equiv ae-4bd+3c^2.$$

To solve this assume that the root is of the form

$$z=\sqrt{p}+\sqrt{q}+\sqrt{r}, \qquad \ldots \quad (3)$$

Squaring this, we get

$$z^2-p-q-r=2(\sqrt{p}\sqrt{q}+\sqrt{q}\sqrt{r}+\sqrt{r}\sqrt{p}).$$

Squaring it again and simplifying, we obtain

$$z^4-2(p+q+r)z^2-8z\sqrt{p}\sqrt{q}\sqrt{r}+(p+q+r)^2-4(pq+qr+rp)=0.$$

Comparing this with (2), we get

$$p+q+r=-3H,\ pq+qr+rp=3H^2-\tfrac{1}{4}a^2I,\ \sqrt{p}\sqrt{q}\sqrt{r}=-\tfrac{1}{2}G.$$

Hence p, q, r are the roots of the equation

$$t^3+3Ht^2+(3H^2-\tfrac{1}{4}a^2I)t-\tfrac{1}{4}G^2=0,$$

which may be written in the form

$$4(t+H)^2-a^2I(t+H)+a^3J=0, \qquad \ldots \quad (4)$$

where
$$J=\frac{1}{a^3}(a^2IH-4H^3-G^2)$$

$$=ace+2bcd-ad^2-eb^2-c^3=\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix}.$$

Putting $t+H=a^2\theta$, we may write (4) as

$$4a^3\theta^3-Ia\theta+J=0. \quad . \quad . \quad . \quad (5)$$

On solving this equation we get three values of θ, which give three values for t, and these are respectively the values of p, q and r. Substitution in (3) then gives the roots.

NOTE. Equation (5) is the reducing cubic obtained in §13·6. Since we have supposed that p, q, r are the roots of the equation (4) in t, and also that $\theta=(t+H)/a^2$, so the roots θ_1, θ_2, θ_3 of (5) will be equal to $(p+H)/a^2$, $(q+H)/a^2$, $(r+H)/a^2$. It may be verified that p, q, r are identical with M_1^2, M_2^2, M_3^2 of the preceding article.

As before, there would be eight combinations of $\sqrt{p}+\sqrt{q}+\sqrt{r}$ on taking either sign with a root; but only those combinations should be taken for which

$$\sqrt{p}\sqrt{q}\sqrt{r}=-\tfrac{1}{2}G.$$

EXAMPLES ON CHAPTER XIII

Solve the equations.

1. $x^3-12x-8=0.$
2. $x^3-6x-4=0.$
3. $x^3-15x-126=0.$ [*Allahabad*, 1960]
4. $x^3-18x-35=0.$
5. $8x^3+24x-63=0.$
6. $9x^3-6x^2+1=0.$
7. $28x^3-9x^2+1=0.$ [*Kashmir*, 1953]
8. $2x^3+3x^2+3x+1=0.$ [*Aligarh*, 1960]
9. $x^3-x^2-3x+6=0.$
10. $x^3+6x^2+3x+2=0.$
11. $x^4+6x^2+8x+21=0.$
12. $x^4+12x-5=0.$ [*I. A. S.*, 1955]
13. $x^4+11x^2+10x+50=0.$ [*Banaras*, 1954]
14. $x^4+4x^3+8x^2+7x+4=0.$
15. $x^4-6x^3+12x^2-20x-12=0.$

16. Find the limits to the values of c such that $x^3-3x+c=0$ may have all its roots real. [*Madras*, 1954]

17. Discuss the nature of the roots of the equation

$$x^3+6x^2+9x+4=0.$$

[Hint. Find the equation of squared differences first.]

18. Show that the two biquadratic equations

$$A_0x^4+6A_2x^2\pm 4A_3x+A_4=0$$

have the same reducing cubic.

19. When a biquadratic has two equal roots, show that its reducing cubic has two equal roots, and conversely.

20. When the roots of the biquadratic are all real, or all imaginary, show that the roots of the reducing cubic are all real; and that when the biquadratic has two real and two imaginary roots, the reducing cubic has two imaginary roots.

CHAPTER XIV

NUMERICAL SOLUTION OF EQUATIONS

14·1. Numerical Equations. We shall now consider practical methods for the determination of the real roots of equations in which the coefficients are given numbers instead of letters. The roots may be integers, fractions or irrational numbers. The roots which are integers can generally be quickly found by trial.

The fractional roots may be changed to integers by multiplying the roots of the given equation by a suitable integer. In this connection the following theorem is useful

An equation in which the coefficient of the first term is unity, and the other coefficients are integers (positive or negative), cannot have a fractional root (but it may have irrational roots).

If possible let one root of the equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n=0,$$

where $p_1, p_2, \ldots$ are integers, be a/b, and suppose that a/b is a fraction in its lowest terms.

Since a/b is a root, we must have

$$\left(\frac{a}{b}\right)^n+p_1\left(\frac{a}{b}\right)^{n-1}+\ldots+p_n=0.$$

Multiplying by b^{n-1} and transposing, we get

$$-a^n/b=p_1a^{n-1}+p_2a^{n-2}b+\ldots+p_nb^{n-1}.$$

Now the left-hand side of this is a fraction (since by hypothesis a is not divisible by b); while the right-hand side is an integer, which is impossible. Hence the proposition.

If an equation contains fractional coefficients when the coefficient of the first term is made unity by division, the roots of the equation can be multiplied by a suitable number, so that the coefficients in the transformed equation are integers. Thus the determination of the fractional roots of a given equation can be reduced to the determination of the integer roots of a transformed equation.

Ex. Remove the fractional coefficients from the equation

$$x^4-\tfrac{5}{6}x^3+\tfrac{5}{12}x^2-\tfrac{13}{900}=0,$$

by multiplying its roots by a suitable number.

The given equation is

$$x^4-\frac{5}{2.3}x^3+\frac{5}{2^2.3}x^2+0.x-\frac{13}{2^2.3^2.5^2}=0.$$

If we multiply the roots by k, the successive coefficients after the first will have to be multiplied by k, k^2, k^3, ... (§ 12·12). Hence, choosing k to be equal to $2\times3\times5$, i.e. 30, will suffice.

Multiplying the roots of the given equation by 30, the transformed equation becomes

$$x^4-25x^3+375x^2-11700=0.$$

14·2. Limits of the roots of equations. In order to reduce the labour involved in searching for roots, it is necessary to be able to obtain some number which possesses the property that all the positive roots of the equation under consideration are less than it. Such a number is called a *superior limit* of the positive roots. The smaller the superior limit is, the better.

Similarly, a superior limit of the negative roots is a negative number numerically greater than all the negative roots.

There are special propositions which enable superior limits to be found, but a little practice and the application of commonsense would enable the student to find a suitable superior limit by merely

grouping the terms, as the following examples will show.

Ex. 1. Find a superior limit of the positive roots of the equation

$$x^4-5x^3+40x^2-8x+23=0.$$

The equation can be written as

$$x^3(x-5)+8x(5x-1)+23=0.$$

The value $x=5$ will make the first term zero, and any greater value of x will make it positive. The value 5 of x or any greater value will make the second term positive. The third term is already positive. Thus $x=5$ and all greater numbers will make $f(x)$ positive. So there cannot be a root greater than 5. Hence 5 is a superior limit of the positive roots.

[6, 7, $6\frac{1}{2}$, or any similar number may also be called a superior limit, but 5 is better than these. Again, substitution will show that 4 is also a superior limit, and this is a better limit than even 5. Thus there cannot be any *one* superior limit.]

Ex. 2. Find superior limits of the positive and negative roots of the equation

$$x^4-2x^3-13x^2+38x-24=0.$$

The equation can be written as

$$x^3(x-5)+x^2(3x-13)+(38x-24)=0. \qquad (1)$$

The first term is zero for $x=5$, while the other terms are positive. Thus the left-hand side is positive for $x \geqslant 5$. Hence 5 is a superior limit of the positive roots.

Changing x to $-x$, we get the equation

$$x^4+2x^3-13x^2-38x-24=0, \qquad (2)$$

or $$x^2(x^2-13)+2x(x^2-22)+6(x-4)=0.$$

We see that the left-hand side is positive for $x \geqslant 5$. Thus 5 is a superior limit of the positive roots of equation (2). Hence -5 is a superior limit of the negative roots of the original equation.

[Instead of (1), we could have arranged the given equation as $$x^3(x-3)+x^2(x-13)+(38x-24)=0,$$

which indicates that 13 is a superior limit of the positive roots. But (1) is a better arrangement as it gives a smaller superior limit.]

14·3. Integer roots. The following procedure will be found convenient when solving a given equation $f(x)=0$.

First find the multiple roots of the equation (§ 11·7) and divide out $f(x)$ by the factors $(x-\alpha)^r\ldots$ corresponding to the multiple roots.

In the new equation obtained, make the coefficient of the first term unity, and the other coefficients integers (§ 14·1). Then the last term (i.e., the absolute term) in the transformed equation must numerically be equal to the product of the roots of this equation. Hence every integer root must be a factor of the last term. Our choice for the positive roots is thus restricted to all the factors of the absolute term which are less than the superior limit. If α, β, γ, ... are possible values of the roots, we should next find which of these satisfy the given equation upon actual substitution, and thus finally find the roots. Also, we can find the negative roots by considering the positive roots of $f(-x)=0$.

Having thus obtained all the rational roots, we can obtain the irrational roots by Newton's or Horner's method, as explained below. The location of these roots may first be obtained by trial by an application of § 11·3(i). If necessary, Sturm's theorem can be applied to obtain the number and location of the real roots.

14·31. Method of Divisors. If α is a root of an equation $f(x)=0$, then $x-\alpha$ is a factor of $f(x)$, i.e., $x-\alpha$ will divide $f(x)$ completely. When testing for possible values of integer

roots, it is, in general, more convenient to divide $f(x)$ by $x-a$ than to evaluate $f(a)$. If the division is carried out in the reverse order, i.e., if $p_n+p_{n-1}x+p_{n-2}x^2+\dots+p_1x^{n-1}+x^n$ is divided by $a-x$, the work is shortened since it can be stopped as soon as fractional coefficients start appearing in the division. A fractional coefficient indicates that the divisor in question does not divide $f(x)$ completely, as may be easily verified by carrying out the division till the end.

Ex. Find the integer roots of the equation

$$x^4-2x^3-13x^2+38x-24=0.$$

By grouping the terms we see (Ex. 2, § 14·2) that all the roots lie between -5 and 5. Since the product of the roots is 24, the following are the possible integer roots :

$$-4, -3, -2, -1, 1, 2, 3, 4.$$

We start with 4. Dividing $-24+38x-13x^2-2x^3+x^4$ by $4-x$, we get

$$\begin{array}{rrrrr} -24 & 38 & -13 & -2 & 1 \\ & -6 & 8 & & \\ \hline & 32 & -5 & & \end{array}$$

Explanation. Here the division has been put down in two lines, and only the coefficients of the terms are written. (The student can verify these by long division). The coefficients in $f(x)$ are written in the first line. Dividing -24 by 4, we get -6. This is written below the next coefficient and added. The sum 32 is again divided by 4 and the quotient written below the next figure. The sum -5 is not divisible by 4. So 4 is not a root and the operation is stopped.

We now divide $f(x)$ by $3-x$. We get

$$\begin{array}{rrrrr} -24 & 38 & -13 & -2 & 1 \\ & -8 & 10 & -1 & -1 \\ \hline & 30 & -3 & -3 & 0 \end{array}$$

So $f(x)$ is completely divisible by $3-x$, giving the quotient $-8+10x-x^2-x^3$. Hence 3 is a root.

Now we proceed to find the roots of $8-10x+x^2+x^3=0$. We divide it by $2-x$ as follows :

$$\begin{array}{rrrr} 8 & -10 & 1 & 1 \\ & 4 & -3 & -1 \\ \hline & -6 & -2 & 0 \end{array}$$

So 2 is also a root. The remaining factor is $4-3x-x^2$, i.e., $(4+x)(1-x)$.

Hence the roots are 4, 2, 1 and -4.

14·4. Newton's method of approximation.* Suppose that one root of the equation $f(x)=0$ is nearly equal to a.

[We can find a by trial : for example if $f(a)$ and $f(a+1)$ are of opposite signs, the root lies between a and $a+1$.]

Let the correct value of the root be $a+y$.

Then $$f(a+y)=0. \quad \ldots \quad (1)$$

But, as we can find by actual substitution (§ 11·6), or by Taylor's Theorem of the Differential Calculus,

$$f(a+y)\equiv f(a)+yf'(a)+\text{higher powers of } y. \quad (2)$$

Since a is an approximate value of the root, y is small. So the second and higher powers of y will be much smaller. Neglecting them, we see that approximately

$$f(a)+yf'(a)=0, \text{ by (1) and (2)},$$

or $$\mathbf{y=-f(a)/f'(a).}$$

This determines y, which added to a gives a better value of the root.

Taking $a+y$ as the new approximate value of the root, the above process can be repeated to find a still better value of the root, and so on.

NOTE. We can obtain $f(a)$ and $f'(a)$ by division : $f(a)$ is the remainder obtained on dividing $f(x)$ by $x-a$; and $f'(a)$

*Named after Isaac Newton (1642-1727), the great English mathematician, who invented the calculus and discovered the famous law of gravitation. The method given here is the modification originally given by Joseph Raphson (1648-1715).

is the remainder obtained on dividing the quotient of the first division again by $x-a$. (See §§ 11·21, 11·6).

Ex. Find to two decimal places by Newton's method of approximation the root of

$$x^3+x^2+x-100=0,$$

which is approximately equal to 4.

Here $$f'(x)=3x^2+2x+1.$$

So $f'(4)=57$. Also $f(4)=-16$.

Therefore, if the root required is $4+y$,

$$y=+16/57 \text{ nearly}=0\cdot3.$$

Hence the root is approximately equal to 4·3.

Now $$f'(4\cdot3)=65\cdot1 \text{ and } f(4\cdot3)=2\cdot3;$$

so the new correction $=-2\cdot3/65\cdot1=-0\cdot036$.

Hence a better value of the root $=4\cdot3-0\cdot036=4\cdot264$.

14·5. Horner's method. Newton's method becomes very tedious after the second or third approximation. Horner's method is also a method of successive approximations, but the labour involved is comparatively small. The essence of the method is the diminution of the roots by the successive digits occurring in the value of the root.

For example, suppose that the required root is 4·2644. The first step will be to determine by trial the first figure 4 in the root and then the roots of the equation will be diminished by this number 4. The roots of the transformed equation are next multiplied by 10 in order to avoid decimals. A root of this new equation will be 2·644. The first figure 2 of this root is now determined, and correspondingly the roots of the equation are diminished by 2. The roots of the resulting equation are again multiplied by 10. The first figure 6 of the new root is next determined and the roots are diminished

by 6; and so on. The root of the original equation can thus be determined to any desired degree of accuracy figure by figure.

The only difficulty which the student will find at every stage is to estimate what the next figure in the root is. The first two or three digits are found by trial and error. After that the coefficients in the transformed equation itself suggest what the next digit is likely to be. These points are explained in the worked out example below.

Ex. Find the positive root of the equation

$$x^3+x^2+x-100=0,$$

correct to four decimal places.* [*Allahabad*, 1960]

By Descartes' rule of signs we see that there can be only one positive root.

By trial we find that $f(4)$ is negative and $f(5)$ is positive, where $f(x)$ denotes the left-hand side of the given equation. Hence the root lies between 4 and 5.

Applying Horner's method, we see that the required root is

$$4{\cdot}2644.$$

The work is shown on the next page.

Explanation. In the first line we write down the coefficients of the given equation, and, after the symbol (, the integral part of the root, viz. 4. Diminish the roots by 4. [In other problems the integral part might involve two or more figures. In such cases also we would diminish the roots by the whole of the integral part.]

The transformed equation, by § 12·14, is

$$x^3+13x^2+57x-16=0.$$

We draw a zig-zag line just above the set of figures 13, 57, -16, viz., the coefficients of the terms in the new equation.

*From Burnside and Panton, *Theory of Equations*, p. 231.

Next we multiply the roots by 10, for which it is only necessary to add one zero at the end of the coefficient of the second term, two zeros at the end of the coefficient of the third term, and so on. We thus get the equation

$$x^3+130x^2+5700x-16000=0. \quad . \quad . \quad . \quad (1)$$

1	1	1	—100	(4.2644
	4	20	84	
	5	21	**−16000**	
	4	36	11928	
	9	**5700**	**−4072000**	
	4	264	3788376	
	130	5964	**−283624000**	
	2	268	256071744	
	132	**623200**	**−27552256**	
	2	8196		
	134	631396		
	2	8232		
	1360	**63962800**		
	6	55136		
	1366	64017936		
	6	55152		
	1372	**64073088**		
	6			
	13780			
	4			
	13784			
	4			
	13788			
	4			
	13792			

One root of this equation must lie between 0 and 9, on account of the way in which it has been formed.

Let $\phi(x)\equiv x^3+130x^2+5700x-16000.$

Then $\phi'(x)=3x^2+2\times130x+5700.$

Thus $\phi(0)=-16000, \ \phi'(0)=5700.$

Therefore, by Newton's method of approximation, the correction to the value 0 of the root of (1) is

$$+16000/5700, \quad . \quad . \quad . \quad (2)$$

which is 2 decimal something. Hence the root of x which is required probably lies between 2 and 3. We shall, therefore, diminish the roots by 2.

Now, if our guess is wrong, i.e., if the approximation given by Newton's method is not very good, and we diminish the roots by a number which is too large, the root of the transformed equation will be negative. Thus if 2·644 is the correct value of the positive root of (1), and we diminish the roots of (1) by 3, there will be no positive root in the transformed equation. This will be indicated by the last term becoming positive; for then, by Descartes' Rule the equation can have no positive root. Hence the roots of (1) *should be diminished by the highest number which will not change the sign of the absolute term.*

If we diminish the root by a number which is too small, the error will become manifest by the next suggested digit in the root being greater than 9.

The third and fourth transformations have been carried out in the same way.

The last figure in the root has been found merely by seeing what the first figure will be in the root of the equation obtained after the fourth transformation.

NOTE. We see from (2) that after every transformation the next digit in the root can be estimated by dividing the absolute term (with its sign changed) by the coefficient of the previous term.

If any digit in the root is zero, a zero should be written there and the coefficients should be multiplied by 10 again by writing down one, two, ... zeros in the second, third, ... coefficients.

Horner's method will readily give fractional roots also.

14·51. Contraction of Horner's Method. If we have to find the root of an equation to seven or eight decimal places by the above method, the coefficients become very large and the method becomes cumbersome. This can be

avoided by contracting the division after a certain stage. Instead of adding a zero at the end of the coefficient of the second term, two zeros at the end of the coefficient of the third term, etc., we strike off the last digit from the coefficient of x, last two digits from the coefficient of x^2, and so on.

The result of this is to multiply the roots by 10 and at the same time neglect those figures which have comparatively little effect in the evaluation of the root. The method is illustrated in the worked out example below.

Ex. Find the positive root of the equation

$$x^3+x^2+x-100=0,$$

correct to nine decimal places.

The first four transformations of this equation will be carried out exactly in the same way as on page 247, giving the root 4·264 to three decimal places. We start below with the coefficients after the fourth transformation, and begin contracting at this stage.

$\cancel{1}$	$137\cancel{9}\cancel{2}$	$6407308\cancel{8}$	-27552256 (4·264,4299732
		552	25631440
		6407860	-1920816
		552	1281688
	$1\cancel{3}\cancel{7}$	$640841\cancel{2}$	-639128
		3	576762
		640844	-62366
		3	57676
	$\cancel{1}$	$64084\cancel{7}$	-4690
		$6408\cancel{4}$	4486
		$640\cancel{8}$	-204
		$64\cancel{0}$	192
		$6\cancel{4}$	-12

The required root is 4·2644 29973.

NOTE. (i) While the cancelled figures are not kept in the products, the carry over due to them has been added. In the first transformation, for example, $137·9\times4=551·6$; so 552 has been added to the third coefficient instead of

137×4, i.e. 448. This ensures greater accuracy with little extra labour. Thus, if the carry over was disregarded we would get 8 in the tenth decimal place instead of 2 which we get above. The correct figure is 1.

(ii) We see that after two contracted transformations, the method simply reduces to contracted division.

(iii) Before starting the contraction the coefficient of x had eight digits. This gave us by the contraction method seven more decimal places in the root. The last of these is in error. So, in general, we should start the contraction process when the coefficient of x contains two more digits than the number of remaining decimal places required in the root.

EXAMPLES ON CHAPTER XIV

Transform into an equation with integral coefficients, and the coefficient of the first term unity:

1. $3x^4-4x^3+4x^2-2x+1=0$.
2. $x^3-\frac{7}{3}x^2+\frac{11}{36}x-\frac{25}{72}=0$.
3. $x^4-\frac{4}{15}x^3-\frac{11}{36}x^2+\frac{31}{300}x+\frac{23}{3600}=0$.

By grouping the terms find upper limits to the positive and negative roots of

4. $x^4-2x^3+3x^2-5x+1=0$.
5. $x^4-2x^3-3x^2+2x-3=0$.
6. $x^5+3x^4+x^3-8x^2-51x+18=0$.
7. $x^4-x^3-2x^2-4x-24=0$.

Find the rational roots of

8. $x^3-9x^2+22x-24=0$.
9. $3x^3-2x^2-6x+4=0$.
10. $x^4+2x^3-7x^2-8x+12=0$.
11. $x^4+9x^3+12x^2-80x-192=0$.
12. $6x^4-7x^3+8x^2-7x+3=0$. [*Utkal*, 1952]
13. $25x^4-70x^3-126x^2+414x-243=0$. [*I.A.S.*, 1960]

14. Apply Newton's method of divisiors to find the integer roots of

$$3x^4-23x^3+35x^2+31x-30=0.$$

[*Delhi (Hons.)*, 1958]

15. Solve $x^4+2x^3+14x+15=0$. [*Gujarat*, 1959]

16. Solve the equation

$$x^4+8x^3+9x^2-8x-10=0.$$ [*Banaras*, 1953]

17. Find by Newton's method the real root of

$$x^3-2x-5=0.$$ [*Aligarh*, 1949]

18. Find by Newton's method the real root of $x^5-4=0$.

19. Find by Newton's method the positive root of

$$2x^3-3x-6=0$$

to four significant figures. [*Aligarh*, 1950]

20. Find by Horner's method the real root of

$$x^3-4x-7=0$$

correct to two decimal places. [*Rangoon*, 1950]

21. Find by Horner's method the positive root of the equation $x^3-6x-13=0$ to four places of decimals [*Allahabad*, 1959]

22. Obtain by Horner's method the root of the equation

$$4x^3-13x^2-31x-275=0$$

which lies between 6 and 7. [*Aligarh*, 1953]

23. Find the positive root of the equation

$$x^3-2x^2-5=0$$

correct to four places of decimals. [*Lucknow*, 1959]

24. Find to 5 decimal places the root of the equation $x^4-12x+7=0$ which lies between 2 and 3. [*I. A. S.*, 1959]

25. Prove that one root of the equation

$$x^4-7x^2+18x-8=0$$

lies between 0 and 1. Find this root to four places oj pɔcimals. [*Lucknow*, 1953]

ANSWERS

Pages 5-6

1. $81x^4-216x^3y+216x^2y^2-96xy^3+16y^4$.
2. $x^{10}+5x^9+10x^8+10x^7+5x^6+x^5$.
3. $\frac{64}{729}x^6-\frac{32}{27}x^4+\frac{20}{3}x^2-20+135/4x^2-243/8x^4+729/64x^6$.
4. $1+8x+20x^2+8x^3-26x^4-8x^5+20x^6-8x^7+x^8$.
5. 82.
6. $60x^4$.
7. $^{18}C_6$.
8. $5670a^4b^4$.
9. $-{}^{11}C_5x^9$; ${}^{11}C_5x^2$.
10. $945a^2$.
11. $\frac{7}{18}$.

Pages 12-13

1. $1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3$; $-1<x<1$.
2. $1+x+\frac{1}{2}(n+1)x^2+\frac{1}{6}(n+1)(2n+1)x^3$; $-1/n<x<1/n$.
3. $4+4x-x^2+\frac{2}{3}x^3$; $-\frac{2}{3}<x<\frac{2}{3}$.
4. $\frac{1}{8}a^{-3/2}+\frac{3}{8}xa^{-5/2}+\frac{15}{16}x^2a^{-7/2}+\frac{35}{16}x^3a^{-9/2}$; $-\frac{1}{2}a<x<\frac{1}{2}a$.
5. $\frac{77}{256}x^{30}$.
6. $-2.5.8\ldots(3r-4)x^r/r!$
7. $\frac{1}{2}(r+1)(r+2)x^r$.
8. $(2r)!(-\frac{1}{4}x)^r/(r!)^2$.
9. $\{1-2^{r+1}(1+r)\}x^r$.
10. $a=\frac{1}{6}$, $b=-\frac{1}{2}$, $c=\frac{4}{3}$.
11. $(-1)^{r/2}$ if r is even, $2(-1)^{(r-1)/2}$ if r is odd.
13. $3.5.7\ldots(2n-1)/2.4.6\ldots(2n-2)$.
14. $4^{1/3}$.
15. $(\frac{9}{4})^{1/3}$.
16. $4(2)^{1/3}-2$.
17. 1.
18. $\frac{1}{2}\{(1-x)^{-1/2}+(1+x)^{-1/2}\}$.

Pages 16-17

1. 4th, $\frac{11583}{1024}$.
2. 4th and 5th, $\frac{7}{144}$.
3. 3rd, $\frac{448}{225}$.
4. 1st, 1.
5. 4th & 5th, $672/5^{10}$.
6. (i) $1-\frac{5}{8}x$, (ii) $2-\frac{1}{4}x$.
7. 9·99333.
8. 0·19960.
9. 1·41421.
10. 1·49535.

Pages 17-19

2. $(-1)^n(3n)!/(2n)!\,n!$ **8.** $1+\frac{13}{6}x+\frac{55}{72}x^2$.

9. $1-\frac{1}{2}x+\frac{3}{8}x^2+\frac{3}{16}x^3$. **19.** $(\frac{3}{2})^{-3/2}$. 20. $\frac{19}{18}$.

21. $3\sqrt{3}$. **22.** $\frac{1}{24}$.

Pages 25-26

1. 1·6487.

2. $2\{1+(2x)^2/2!+(2x)^4/4!+\ldots+(2x)^{2r}/(2r)!+\ldots\}$.

3. $(-1)^n(n+1)^2/n!$ **4.** $2e$.

5. $\frac{1}{8}(e^4-e^{-4})$. **6.** $e-1$.

7. $1-2/e$. **8.** $(x^3+6x^2+7x+1)e^x$.

9. $2e-\frac{7}{2}$. **10.** $\frac{17}{8}e$.

Pages 30-31

1. $\log 2+\frac{1}{2}x-\frac{1}{2}(\frac{1}{2}x)^2+\frac{1}{3}(\frac{1}{2}x)^3-\ldots+(-1)^{r-1}(\frac{1}{2}x)^r/r+\ldots$.

2. $3x-5x^2/2+9x^3/3-17x^4/4+\ldots+(-1)^{r-1}(2^r+1)x^r/r+\ldots$.

3. $-x+x^2/2+x^3/6+\ldots+x^r/r(r-1)+\ldots$.

4. $x+3x^2/2+x^3/3+3x^4/4+\ldots+\{2+(-1)^r\}x^r/r+\ldots$.

7. $\log_e \frac{4}{3}$. **9.** $y+\frac{1}{2}y^2+\frac{1}{3}y^3+\ldots$.

10. 0·84510; 1·04139; 1·11394.

12. $\log_3 e\,.\log_e 2$; $\frac{1}{4}(x-x^{-1})\log\{(1+x)/(1-x)\}+\frac{1}{2}$.

Pages 31-34

1. $e^a b^n/n!$ **7.** 0·002000.

15. $1/\log_e 2$. **16.** $\frac{3}{2}e$.

22. $2\Sigma\{x^{6r-5}/(6r-5)-2x^{6r-3}/(6r-3)+x^{6r-1}/(6r-1)\}$,
starting from $r=1$.

Pages 38-39

3. a^3+2b^3. **5.** x^3-1 if $x>1$; x^2-x if $x<1$.

7. x must be less than 2.

Pages 44-45

17. 2; 3. **18.** $6^3\times 8^4$.

19. $3/(2\times 5^5)$. **20.** 9, when $x=1$.

PAGES 52-54

1. (i) When $x+y>0$; (ii) If $x^2>1$, then y^2 must <1; if $x^2<1$, then y^2 must >1.

12. $4^4\times 5^5$, when $x=3$. **13.** $(\frac{3}{5})^{1/2}(\frac{2}{5})^{1/3}$ when $x=\frac{3}{5}$.

PAGE 61

1. $2/(x-1)+3/(x+4)$. **2.** $5/(2x-1)-4/(3x-1)$.

3. $1+1/(x-1)-1/(x+1)$. **4.** $3/(x-1)-4/(2x-1)$.

5. $2/(x-1)+3/(x-2)-4/(x-3)$.

6. $(6-\sqrt{6})/4(x+1-\sqrt{6})+(6+\sqrt{6})/4(x+1+\sqrt{6})-1/(x-3)$.

7. $1+\Sigma\{a^3/(a-b)(a-c)(x-a)\}$.

8. $3/4(x+1)+1/4(x-1)+1/2(x-1)^2$.

9. $x-1-3/x+1/(x-1)+4/(x+1)+1/(x+1)^2$.

10. $2/(x+1)+3/(x+1)^2-6/(3x+2)$.

PAGE 65

1. $3/(x-1)+1/(x-1)^2-7/(x-1)^3+5/(x-1)^4$.

2. $128/125(x+4)+122/125(x-1)+28/25(x-1)^2+2/5(x-1)^3$

3. $1/(x-1)-1/(x+1)+3/(x+1)^2-3/(x+1)^3+2/(x+1)^4$.

4. $1/16(x+1)+15/16(x-1)+17/8(x-1)^2+7/4(x-1)^3$
$+1/2(x-1)^4$.

5. $-1/27(x-1)+2/9(x-1)^2+1/27(x+2)-1/9(x+2)^2$.

6. $11/25(x-1)+2/5(x-1)^2-(11x-4)/25(x^2+4)$.

7. $3/5(x+1)+(7x+8)/5(x^2+4)$.

8. $1/(x-1)-3/(x+1)+(2x+1)/(x^2+x+1)$.

9. $(3x+2)/(x^2+3)-(2x+1)/(x^2+2)$.

10. $1/2(x+1)^2-5/2(x^2+1)+2/(x^2-x+1)$.

PAGES 67-68

1. $-1/x+1/(x-1)+1/(x+1)$.

2. $\Sigma a/(a-b)(a-c)(x-a)$.

3. $1/3(2x-1)-5/3(x-2)-4/(x-2)^2$.

4. $x-2-17/16(x-3)+17/16(x+1)-11/4(x+1)^2$.

5. $7/32(x+1)-21/32(3x-1)+21/8(3x-1)^2-3/2(3x-1)^3$.

6. $-1/(x-1)+2/(x-1)^3+1/(x-1)^4+x/(x^2-x+1)$.

7. $-1/2(x-1)-1/2(x-1)^2+3/5(x-2)-(x+2)/10(x^2+1)$.

8. $(x-5)/27(x^2+2)-1/27(x+2)+10/9(x+2)^2-4/3(x+2)^3$.

9. $(x+\sqrt{2})/2\sqrt{2}(x^2+x\sqrt{2}+1)-(x-\sqrt{2})/2\sqrt{2}(x^2-x\sqrt{2}+1)$.

10. $1-x/2(x^2+x+1)+x/2(x^2-x+1)$.

11. $1-(-\frac{2}{3})^{r+1}$.

12. $1+(-1)^{r-1}-2^{r+2}$.

13. $3+4(-1)^{r/2}$ if r is even, $3(-1)^{(r+1)/2}-3$ if r is odd.

14. $(-1)^r\{3r+5-3(\frac{3}{2})^r\}$.

15. $4^{r-1}(11r+12)$.

16. $(-1)^r(3.2^{-r}-\frac{11}{12}.3^{-r}-\frac{7}{4}+\frac{3}{2}r)$.

17. $\Sigma a^{r+2}/(a-b)(a-c)$.

18. $-4/9(x+2)+4/9(x-1)-1/3(x-1)^2$.

19. $1/(1-x)(1-x^2)$.

20. $(1-a)^{-2}\{-(1+x)^{-1}+(1+ax)^{-1}+(1+a^n x)^{-1}-(1+a^{n+1}x)^{-1}\}$.

Pages 75-76

1. $u_n-3u_{n-1}+3u_{n-2}-u_{n-3}=0$.

2. $u_n-7u_{n-1}+12u_{n-2}=0$.

3. $a_n-4a_{n-1}+4a_{n-2}=0$.

5. $a_n-4a_{n-1}+3a_{n-2}=0$; $(1-2x)/(1-4x+3x^2)$.

6. $a_n-2a_{n-1}-a_{n-2}=0$; $1/(1-2x-x^2)$.

7. $a_n+a_{n-1}-2a_{n-2}=0$; $(2+x)/(1+x-2x^2)$.

8. $a_n-3a_{n-1}+3a_{n-2}-a_{n-3}=0$; $(2-x+x^2)/(1-x)^3$.

9. $2^{n+1}-3^{n-1}$.

10. $\frac{8}{5}(-1)^n+2^{2n-3}/5$.

11. $(4^{n-1}+3^{n-1})x^{n-1}$.

12. $\{2^{n-1}-4(-1)^n\}x^{n-1}$.

13. $(1-x^n)(x+2)/(1-x)^2-3nx^n/(1-x)$.

14. $2(1-3^n x^n)/(1-3x)-3(1-2^n x^n)/(1-2x)$.

15. $2^{n+2}-2n-4$.

16. $\frac{1}{9}[13(2^n-1)-3n \,.\, 2^{n-1}-\frac{1}{2}\{1-(-1)^n\}]$.

17. $\frac{1}{7}[\frac{8}{3}(3^n-1)-\frac{11}{5}\{(-4)^n-1\}]$.

18. $\frac{1}{4}\{(-x)^{n-1}+3(3x)^{n-1}\}$; $\frac{1}{4}\{n+\frac{3}{4}+\frac{1}{4}(-3)^{n+1}\}$.

19. $(1-8x)/(1-x-6x^2)$; $2(-2x)^{n-1}-(3x)^{n-1}$;
$-2\{(-2x)^n-1\}/(2x+1)-\{(3x)^n-1\}/(3x-1)$.

Page 83

1. $2+\frac{1}{11+}\,\frac{1}{1+}\,\frac{1}{31}$. **2.** $\frac{1}{5+}\,\frac{1}{13+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{3}$.

3. $3+\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{8+}\,\frac{1}{5+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2}$.

4. $\frac{1}{3+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{3+}\,\frac{1}{4}$.

5. 2/1, 11/5, 13/6, 37/17, 346/159, 1075/494.

6. 1/2, 2/5, 7/17, 9/22, 25/61, 84/205, 193/471.

9. $1+\frac{1}{a-2+}\,\frac{1}{1+}\,\frac{1}{2(a-1)}$.

10. $\frac{1}{a+}\,\frac{1}{(a+1)+}\,\frac{1}{(a+2)+}\,\frac{1}{(a+3)}$; $\frac{a^2+3a+3}{a^3+3a^2+4a+2}$.

Pages 87-88

1. (i) $1/296<e_1<1/232$; (ii) $1/103\times 964<e_2<1/103\times 861$.

3. 151/115. **5.** 157/225.

Page 93

1. $\frac{1}{4}(9+\sqrt{5})$. **2.** $\frac{1}{2}(\sqrt{15}-1)$. **3.** $\sqrt{\frac{13}{2}}-2$.

6. $2+\frac{1}{4+}\dots$ **7.** $2+\frac{1}{4+}\,\frac{1}{4+}\dots$

8. $3+\frac{1}{6+}\dots$ **9.** $4+\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{8+}\dots$

12. $\{3^{n+1}+3(-1)^{n+1}\}/\{3^{n+1}-(-1)^{n+1}\}$.

Pages 93-95

1. $\frac{1}{7}(\sqrt{37}-4)$.

3. First 3 convgts are $(n-1)/1$, $n^2/(n+1)$, $(n^3-n^2+n-1)/n^2$.

Page 104

1. 1. **2.** $\frac{1}{2}\times 3^{1/3}$. **3.** 0.

4. ∞. **5.** 0. **6.** 0.

8. (i) $\frac{1}{2}$; (ii) 1. **9.** Convergent. **10.** Divergent.

11. Convergent. **12.** Convergent.

Pages 110-111

1. Divergent. **2.** Divergent. **3.** Convergent.

4. Convergent. **5.** Con. **6.** Con. if $x<1$, div. if $x\geqslant 1$.

7. Con. if $x<1$, div. if $x\geqslant 1$. **8.** Con. if $x<1$, div. if $x\geqslant 1$.

9. Con. if $x<1$, div. if $x\geqslant 1$. **10.** Con. if $x<1$, div. if $x\geqslant 1$.

Pages 115-116

1. Div. **2.** Div. **8.** Con.

4. Div. **5.** Con. if $p>2$, div. if $p\leqslant 2$.

6. Con. **7.** Div. 8. Con.

9. Con. **10.** Con. if $x\leqslant 1$, div. if $x>1$.

11. Con. if $x\leqslant 1$, div. if $x>1$. **12.** Con. if $x<1$, div. if $x\geqslant 1$.

Page 121

1. Con. if $a\leqslant 0$, div. if $a>0$. **2.** Con.

3. Con. if $x^2\leqslant 1$, div. if $x^2>1$.

4. Con. if $x^2\leqslant 1$, div. if $x^2>1$.

5. Con. if $x<1$, div. if $x\geqslant 1$.

6. Con. if $x<1/e$, div. if $x\geqslant 1/e$.

Page 126

1. Con. **2.** Div. **3.** Con. if $b-a>1$, div. if $b-a\leqslant 1$.

4. Div. **5.** Con. if $x<1$, div. if $x\geqslant 1$.

6. Con. if $x<1$, div. if $x>1$;
when $x=1$, con. if $\gamma-\alpha-\beta>0$, div. if $\gamma-\alpha-\beta\leqslant 0$.

Page 132

1. Convergent. **2.** Convergent. **3.** Oscillatory.

4. Convergent if $x\leqslant 1$; oscillatory if $x>1$.

5. (*a*) Oscillatory; (*b*) Con. if $x \leqslant 1$; oscillatory if $x > 1$.
6. (i) No; (ii) Yes; (iii) Yes; (iv) Absolutely con. if $x < 1$; not con. if $x \geqslant 1$.

Pages 138-140

1. Divergent. 2. Convergent.
3. Convergent if $p > 1$; div. if $p \leqslant 1$. 4. Convergent.
5. Divergent. 6. Divergent.
7. Convergent if $p > 2$; divergent if $p \leqslant 2$.
8. Convergent. 9. Convergent. 10. Convergent.
11. Convergent. 12. Convergent. 13. Divergent.
14. Divergent. 15. Divergent. 16. Divergent.
17. Divergent if $x \leqslant a$, convergent if $x > a$, supposing $a > 0$.
18. Con, if $x \neq 1$. 19. Con. if $x < 1$; div. if $x \geqslant 1$.
20. Con. if $x < 1$; div. if $x \geqslant 1$. 21. Con. if $x \leqslant 1$; div. if $x > 1$.
22. Con. if $x \leqslant 1$; div. if $x > 1$. 23. Con. if $x < e$; div. if $x \geqslant e$.
24. Convergent if $x < 1/e$; divergent if $x \geqslant 1/e$.
25. Convergent if $x < 1$; divergent if $x \geqslant 1$; the result holds for all values of q, positive, zero, or negative.
26. (*a*) Convergent, (*b*) Convergent. 27. Convergent.

Pages 151-153

1. $abc + 2fgh - af^2 - bg^2 - ch^2$. 2. $3abc - a^3 - b^3 - c^3$.
3. 3. 4. 0. 5. 0.
6. 6. 14. $x = 4$. 15. $x = -10/89$.

Pages 157-160

1. 0. 2. 0. 3. $a(x-a)^3$.
4. $-(a+b+c)(b+c-a)(c+a-b)(a+b-c)$.
5. $abcd(1 + 1/a + 1/b + 1/c + 1/d)$.
6. $(a-b)(b-c)(a-c)(a-d)(b-d)(c-d)$.
7. 0. 8. $(a-b)(b-c)(c-a)(ab+bc+ca)$.

9. $0, \pm\sqrt{\{\frac{3}{2}(a^2+b^2+c^2)\}}$.

10. $x=1, 2, -3$.

19. $\begin{vmatrix} b^2+c^2 & ab & ac \\ ba & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix}$.

21. $4a^2b^2c^2$.

22. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix} = (a^3+b^3+c^3-3abc)^2$.

Pages 170-171

1. (i) $[5\ \ 7\ \ 9]$, (ii) $\begin{bmatrix} 0 & -1 \\ 5 & -4 \end{bmatrix}$, (iii) $\begin{bmatrix} -1 & 0 & 18 \\ 4 & 19 & -21 \end{bmatrix}$.

2. $\begin{bmatrix} 3 & -2 \\ 5 & -5 \\ 7 & -8 \end{bmatrix}$, No.

3. $\begin{bmatrix} 4 & 4 & -2 \\ 1 & 1 & 10 \\ -1 & 5 & -4 \end{bmatrix}$, $\begin{bmatrix} -5 & 0 & 7 \\ -4 & 5 & 3 \\ 5 & 4 & 1 \end{bmatrix}$.

4. (i) $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$; (ii) $[30]$, $\begin{bmatrix} 5 & 10 & 15 & 20 \\ 4 & 8 & 12 & 16 \\ 3 & 6 & 9 & 12 \\ 2 & 4 & 6 & 8 \end{bmatrix}$.

5. $\begin{bmatrix} -6 & 1 & 2 \\ 5 & 4 & 4 \\ 2 & 8 & -3 \end{bmatrix}$

6. $\begin{bmatrix} 1 & 2 & -3 \\ -2 & 3 & 1 \\ 3 & -1 & 2 \end{bmatrix}$, $\begin{bmatrix} 14 & -7 & 1 \\ -7 & 14 & -5 \\ 1 & -5 & 14 \end{bmatrix}$.

7. (i) $\begin{bmatrix} 50 & 19 \\ -284 & -12 \\ 167 & 21 \end{bmatrix}$, (ii) $\begin{bmatrix} 9 & 6 \\ -18 & -12 \\ 27 & 18 \end{bmatrix}$.

8. $\begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix}$, $\begin{bmatrix} -7 & 30 \\ 60 & -67 \end{bmatrix}$, $\begin{bmatrix} \frac{3}{11} & \frac{2}{11} \\ \frac{4}{11} & -\frac{1}{11} \end{bmatrix}$.

11. $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 4 & -6 \\ -1 & 5 & -7 \end{bmatrix}$, $\begin{bmatrix} 2 & 6 & 4 \\ 21 & -7 & -8 \\ -18 & 6 & 4 \end{bmatrix}$, $\begin{bmatrix} \frac{1}{10} & \frac{3}{10} & \frac{1}{5} \\ \frac{21}{20} & -\frac{7}{20} & -\frac{2}{5} \\ -\frac{9}{10} & \frac{3}{10} & \frac{1}{5} \end{bmatrix}$.

Pages 176-177

1. $\frac{1}{6}\{(2n-1)(2n+1)(2n+3)+3\}$.

2. $\frac{1}{12}\{(3n-1)(3n+2)(3n+5)(3n+8)+80\}$.

3. $\frac{1}{16}\{(4n-3)(4n+1)(4n+5)(4n+9)+135\}$.

4. $\frac{1}{10}\{(2n-1)(2n+1)(2n+3)(2n+5)(2n+7)+105\}$.

5. $\frac{1}{5}n(n+1)(n+2)(n+3)(4n+11)$.

6. $\frac{1}{4}n(n+1)(n+6)(n+7)$.

7. $\frac{3}{4}n(n+1)(3n-2)(3n+1)$.

8. $n/(2n+1)$, $\frac{1}{2}$.

9. $n(5n+13)/12(n+2)(n+3)$, $\frac{5}{12}$,

10. $n(n+2)/3(2n+1)(2n+3)$, $\frac{1}{12}$.

11. $n(2n+7)/45(4n+5)(4n+9)$, $1/360$.

12. $\frac{1}{90}-1/6(2n+1)(2n+3)(2n+5)$, $\frac{1}{90}$.

13. $n(4n+5)/3(2n+1)(2n+3)$, $\frac{1}{3}$.

14. $\dfrac{29}{36}-\dfrac{6n^2+27n+29}{6(n+1)(n+2)(n+3)}$, $\dfrac{29}{36}$.

15. $3\log_e 2-1$.

16. $\dfrac{(n+1)!}{(2-a)\,.\,a(a+1)\ldots(a+n-1)}-\dfrac{1}{2-a}$.

17. $\{5\,.\,7\,.\,9\ldots(2n+5)/6\,.\,8\ldots(2n+4)\}-5$.

18. $\{4\,.\,7\,.\,10\ldots(3n+4)/2\,.\,5\,.\,8\ldots(3n+2)\}-2$.

Pages 178-183

2. $2^{1/3}-1$.

5. $x^2-x^3+\frac{11}{12}x^4-\ldots+(-1)^n 2\{1+\frac{1}{2}+\frac{1}{3}+\ldots$
$+1/(n-1)\}x^n/n+\ldots$.

20. $10/9(x+2)-4/3(x+2)^2-(x+4)/9(x^2+2)$.

21. $24/(x-2)^4+12/(x-2)^3+6/(x-2)^2+1/(x-2)-1/(x+1)$.

22. $1/4(x-1)-(x+1)/4(x^2+1)-(x-1)/2(x^2+1)^2$.

23. $1+\Sigma a^3/(a-b)(a-c)(x-a)$.

26. $A=x$, $B=x+1$.

27. $(bc+1)/(ab+1)$.

28. $2\{(3+\sqrt{13})^n-(3-\sqrt{13})^n\}/\{(3+\sqrt{13})^{n+1}-(3-\sqrt{13})^{n+1}\}$.

29. 185, 96.

30. Divergent.

31. Convergent.

32. Convergent.

33. Oscillatory.

34. Convergent if $x<1$, divergent if $x\geqslant 1$.

35. Convergent if $x<1$, divergent if $x>1$. When $x=1$, convergent if $b>a+d$ and divergent if $b\leqslant a+d$.

36. Convergent if $x \leqslant 1$, divergent if $x > 1$.
37. Convergent if $x < 1/e$, divergent if $x \geqslant 1/e$.
40. $(a-1)^6$.
45.
$$\begin{vmatrix} 0 & b & c \\ c & a & 0 \\ b & 0 & a \end{vmatrix}^2 = 4a^2b^2c^2.$$
46. $x=1, y=-2, z=3$.
47. $x=k(k-c)(k-b)/a(a-c)(a-b)$, etc.
50. $2-3(\frac{2}{3})^{n+1}/(n+1)$.

Pages 194-195

1. $3x^3+4x^2+22x+77$, 170.
2. $x^4-4x^3+8x^2-16x+32$, -61.
3. -1, 4.
4. $-1 \cdot 5232$.
6. $3x^3+7x^2-11x-15=0$.
7. $x^4+5x^2-22x-10=0$.
8. 1, 3, 5, 7.
9. $2, -1, \frac{1}{2} \pm \frac{5}{2}i$.
10. $-\frac{3}{2}, -\frac{1}{3}, 2 \pm \sqrt{3}$.

Pages 199-200

1. 2, 2, -1.
2. 1, 3, 5.
3. -1, 2, 5.
4. 6, 2, $\frac{2}{3}$.
5. $1, \frac{1}{2}, \frac{1}{3}$.
6. -2, 3, 6.
7. $2, -1, -\frac{3}{2}$.
8. -2, 3, 4.
9. $\pm\sqrt{3}, 1 \pm i\sqrt{6}$.
10. $-q/r$.
11. $-p^3+3pq-3r$.
12. q^2-2pr.
13. $pq/r-3$.
14. $r-pq$.
15. $pr-4s$.

Page 202

1. $-\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}$.
2. 2, 2, -1, -3.
3. 1, 1, 1, -3.
4. 1, 1, 2, 3.
5. $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -2$.
6. $-2, (1 \pm i\sqrt{3})/2, (1 \pm i\sqrt{3})/2$.
7. $G^2+4H^3=0$.
10. 10.
11. 123.
12. $5cd/a^2$.

PAGES 206-207

1. One real root between 2 and 3.
2. Two roots between $-1, 0$; and one between $2, 3$.
3. Two real roots in the intervals $-1, 0$ and $1, 2$.
4. All roots real; one each in intervals $-3, -2$; $-1, 0$; and two in the interval $2, 3$.
5. One real triple root and a real positive root.
6. Two real positive roots and two imaginary roots.
7. Four imaginary and one real negative root.

PAGES 210-211

1. One positive, one negative and two imaginary roots.
2. One positive, one negative and two imaginary roots.
4. $-1\pm\sqrt{2},\ -1\pm\sqrt{(-1)}$.
5. $-2, 1, 4$.
6. $-3, 7, 9$.
7. $-4, -1, 2, 5$.
8. $(3\pm\sqrt{5})/2, (-1\pm\sqrt{5})/2$.
9. $-2, (1\pm i\sqrt{15})/2$.
10. $4, 2, \frac{4}{3}$.
11. $1, 1, 2, 2, -3$.
14. $a^2d+2b^3=3abc$.
16. $3r-pq$.
17. $5, 43$.
18. 5.

PAGES 221-223.

1. $x^7+3x^5+x^3+x^2+7x-2=0$.
2. $x^3+6x^2-36x+27=0$.
3. $x^4-x^3+6x^2-6x-4=0$.
4. $x^3+x+3=0$.
5. $x^5+15x^4+97x^3+332x^2+588x+434=0$.
6. $x^3-8x^2+19x-15=0$.
7. $x^3-8x-15=0$.
8. $x^4-24x^2+65x-55=0$.
9. $x^4-4x^2+1=0$.
10. $x^4+8x^3-111x-196=0$, or $x^4-8x^3+17x-8=0$.
11. $x^3+33x^2+12x+8=0$.
12. $x^3+qx-r=0$.
13. $x^3-q^2x^2-2qr^2x-r^4=0$.
14. $rx^3+q^2x^2-2qrx+r^2=0$.
15. $x(rx+q)^2=r$.
16. $rx^3-qx^2-1=0$.
17. $rx^3+q(1-r)x^2+(1-r)^3=0$.

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18. $x^3-2qx^2+(q^2+pr)x+r(r-pq)=0.$

19. $(pq-r)x^3-(p^3-2pq+3r)x^2+(pq-3r)x-r=0.$

20. $r^2x^3+pr(r-1)x^2+q(r-1)^2x+(r-1)^3=0.$

21. $3x^3-11x^2+9x-2=0.$

23. $1, (-7\pm i3\sqrt{3})/2.$

24. $x^3-42x^2+441x-400=0.$

25. $x^3-30x^2+225x-68=0.$

26. $2, \frac{1}{2}, i, -i.$

27. $3\pm2\sqrt{2}, 2\pm\sqrt{3}.$

28. $-1, 2\pm\sqrt{3}, (1\pm i\sqrt{15})/4.$

29. $1, 2, \frac{1}{2}, 2\pm\sqrt{3}.$

30. $\pm1, 2, \frac{1}{2}, (5\pm i\sqrt{11})/6.$

31. $\frac{1}{4}\{3+\sqrt{5}\pm i\sqrt{(10+2\sqrt{5})}\}, \frac{1}{4}\{3-\sqrt{5}\pm i\sqrt{(10-2\sqrt{5})}\}.$

32. $16x^4+24x^3+4kx^2+24x+16=0;\ 2, 2, (1\pm i\sqrt{15})/2.$

Pages 237-238.

1. $4\cos 20°, 4\cos 140°, 4\cos 260°.$

2. $2\sqrt{2}\cos 15°, 2\sqrt{2}\cos 135°, 2\sqrt{2}\cos 255°.$

3. $6, -3\pm i2\sqrt{3}.$

4. $5, \frac{1}{2}(-5\pm i\sqrt{3}).$

5. $\frac{3}{2}, \frac{1}{4}(-3\pm i5\sqrt{3}).$

6. $-\frac{1}{3}, \frac{1}{6}(3\pm i\sqrt{3}).$

7. $-\frac{1}{4}, \frac{1}{7}(2\pm i\sqrt{3}).$

8. $-\frac{1}{2}, (-1\pm i\sqrt{3})/2.$

9. $-2, \frac{1}{2}(3\pm i\sqrt{3}).$

10. $-2-k\sqrt[3]{3}-k^2\sqrt[3]{9},$ where $k=1, \omega, \omega^2.$

11. $1\pm i\sqrt{6}, -1\pm i\sqrt{2}.$

12. $-1\pm\sqrt{2}, 1\pm2i.$

13. $1\pm3i, -1\pm2i.$

14 $\frac{1}{2}(-1\pm i\sqrt{3}), \frac{1}{2}(-3\pm i\sqrt{7}).$

15. $2\pm\sqrt{6}, 1\pm i\sqrt{5}.$

16. $-2\leqslant c\leqslant 2.$

17. Two equal roots.

Pages 250-251.

1. $x^4-4x^3+12x^2-18x+27=0.$

2. $x^3-14x^2+11x-75=0.$

3. $x^4-8x^3-275x^2+2790x+5175=0.$

4. $2, 0.$

5. $3, -2.$

6. $3, -5.$

7. $4, -2.$

8. $6.$

9. $\frac{2}{3}.$

10. $1, 2, -2, -3.$

11. $-4, -4, -4, 3.$

12. $\frac{1}{2}$, $\frac{2}{3}$.

13. 1, 3.

14. 3, 5, -1.

15. -1, -3, $1 \pm 2i$.

16. 1, -1, $4 \pm \sqrt{6}$.

17. 2·0946.

18. 1·3195.

19. 1·784.

20. 2·59.

21. 3·1768.

22. 6·25.

23. 2·6907.

24. 2·04728.

25. 0·5616.